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PI STABILIZATION FOR TIME DELAY SYSTEM

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ABSTRACT: *In this paper, we propose a solution for the stabilizing problem of second order delay plant by Proportional-Integral (PI) controller. An extension of the Hermite-Biehler theorem, which is applicable to quasi-polynomials, is used to seek the stability region of the controller and the computation of its optimum parameters via Genetic Algorithms. Moreover we compare the stabilities regions and different optimization performances obtained for a system which can be modeled by first and second order delay system.*.

KEYWORDS: *second order delay system, PI controller, Hermit-Biehler theorem, stability region, genetic algorithm*

1 INTRODUCTION

Abstract—In this paper, we propose a solution for the stabilizing problem of second order delay plant by Proportional-Integral (PI) controller. An extension of the Hermite-Biehler theorem, which is applicable to quasi-polynomials, is used to seek the stability region of the controller and the computation of its optimum parameters via Genetic Algorithms. Moreover we compare the stabilities regions and different optimization performances obtained for a system which can be modeled by first and second order delay system.

Keywords— second order delay system, PI controller, Hermit-Biehler theorem, stability region, genetic algorithm

1. Introduction

Time delay system are often encountered in various engineering systems such as electrical and communication network, chemical process, turbojet engine, nuclear reactor, hydraulic system; it is frequently a source of instability, oscillation and poor performance in many dynamic systems. Furthermore, delay makes system analysis and control design much more complex [15]. Therefore the problem of stability, stabilization and control of time delay system has been the subject of considerable research. The PID controller remains one of the most popular control method used in process industry, it success is mainly due to its simple structure and it performs well for a wide range of processes [14]. O'dwyer presents a survey of tuning rules for PI/PID control for delayed system [2]. Another complex structure with additional parameters present better performance such as PI predictive controller for first order delay system [17]. In the work presented in [12], the authors have developed a graphical approach, based on D-partition method [10], to describe the borders of absolute and relative stability regions in the parameter space. A unified approach and Delta operator are used to obtain unified stability boundaries for PI, PD and PID controller for an arbitrary order delay system where the boundaries can be found when only the frequency response and not the parameters of the plant are known. Lass et al [11] consider a design of PID controller for system with varying time delay using multi-objective optimization. Recent studies have made use of the generalization of Hermite-Biehler Theorem to compute the set of all stabilizing PID controllers for given linear time invariant plant [9]. Roy and Iqbal have explored PID tuning of first order delay system using a first order Padé approximation and the Hermite-Biehler framework [1].The characterization of the set of all stabilizing P/PI/PID parameters
using a version of the Hermit-Biehler Theorem for the first order delay system is presented in [5,6,7]. The similar approach is applied for determining the stabilizing feedback gains for second order system with time delay [9].

In our work we extend the Silva et al approach to a second order delay system using PI controller which is considered as more complicated than the P stabilization problem [9]. Also, we look for optimum regulator under different criteria using the Genetic Algorithms (GA). Finally, we have simulated this approach to vacuum distillation process, which can be modeled by a first and a second order delay system, in order to compare the stabilities regions and different non-convex performances (ISE, IAE, ITAE and ITSE) optimized by GA.

This paper is organised as follows: in section 2 we state our main results, the preliminary knowledge and the problem formulation are given in section 3. Section 4 is devoted to stabilization problem for second order delay system controlled via PI controller. In order to obtain optimal regulator in the zone of stability, a description of the genetic algorithms is presented in section 5. Section 6 is reserved for simulations.

2. Main Results

The set's characterization of all stabilizing PI/PID parameters using a version of the Hermit-Biehler Theorem for first order delay system is presented in [5, 6, 7, 8]. However, these results are not applicable to the second order delay system.

In this section, the stabilizing problem of PI controller for second order delay system is analyzed using the extended Hermit-Biehler theorem for quasi-polynomials which is presented in section 3. The proof of these results is given in section 4.

The problem considered in this paper involves PI controller design for feedback system in which the plant and the controller transfer function are given by:

$$G(s) = \frac{K}{s^2+a_1s+a_0} e^{-Ls}, C(s) = K_p + \frac{K_i}{s} \qquad (1)$$

Where K is the static gain of the plant, L is the time delay and a_0, a_1 are the plant parameters which are always positive.

Our objective is to determine analytically the region in the (K_p, K_i) parameter space for which the closed-loop system is stable.

Next we state our main results of this paper.

Theorem 1

Under the above assumptions on K, L, a_0 and a_1, the range of K_p values for which a solution exists to the PI stabilization problem of a open-loop stable plant with transfer function $G(s)$ is given by:

$$-\frac{a_0}{K} < K_p < \frac{1}{K}(a_1 \frac{\alpha}{L} \sin(\alpha) - \cos(\alpha)(a_0 - \frac{\alpha^2}{L^2})) \qquad (2)$$

Where α is the solution of the equation $tan(\alpha) = \dfrac{\alpha(2+a_1L)}{(\alpha^2 - a_1L - a_0L^2)}$ in the interval $\left[0, \pi\right]$.

3. Problem Statement

In time delay systems the characteristic equations usually has the form [8, 16]:

$$\delta(s) = d(s) + e^{-L_1s} n_1(s) + e^{-L_2s} n_2(s) + ... + e^{-L_ms} n_m(s)$$

(3)

Where: $d(s)$ and $n_i(s)$ are polynomials with real coefficients and L_i represent time delays. These characteristic equations are recognized as quasi-polynomials. Based on Pontryagin's results [8] an extension of the Hermit-Biehler Theorem can be used to study the stability of certain class of quasi-polynomials. We can consider the quasi-polynomials $\delta^*(s)$ described as follow:

$$\delta^*(s) = e^{sL_m} \delta(s) \qquad (4)$$

$$\delta^*(s) = e^{sL_m} d(s) + e^{s(L_m - L_1)} n_1(s) + e^{s(L_m - L_2)} n_2(s) + ... + n_m(s)$$

by using the following assumptions:

(A_1) $\deg(d(s)) = n$ *and* $\deg(n_i(s)) < n$ *for* $i = 1, 2, ..., m$

(A_2) $L_1 < L_2 < ... < L_m$

(5)

Since e^{sL_m} does not have any finite zeros in the complex plane, the zeros of $\delta(s)$ are identical to those of $\delta^*(s)$. If $\delta^*(s)$ does not have a principal term, then it has an infinity roots with positive real parts [8, 16]. We say that $\delta^*(s)$ has a principal term if the coefficient of the term containing the highest powers of s and e^s is nonzero.

The stability of the system with characteristic equation (3) is equivalent to the condition that all the zeros of $\delta^*(s)$ must be in the open left half of the complex plane. We said that $\delta^*(s)$ is Hurwitz or is stable. The following theorem gives a necessary and sufficient condition for the stability of $\delta^*(s)$ [8].

Theorem 2 [8]

Let $\delta^*(s)$ be given by (3), and write:

$$\delta^*(j\omega) = \delta_r(\omega) + j\delta_i(\omega) \qquad (6)$$

where $\delta_r(\omega)$ and $\delta_i(\omega)$ represent respectively the real and imaginary parts of $\delta^*(j\omega)$.

Under conditions A_1 and A_2, $\delta^*(s)$ is stable if and only if:

1: $\delta_r(\omega)$ and $\delta_i(\omega)$ have only simple, real roots and these interlace,

2: $\delta'_i(\omega_0)\delta_r(\omega_0) - \delta_i(\omega_0)\delta'_r(\omega_0) > 0$ for some ω_0 in $\left[-\infty, +\infty\right]$.

where $\delta'_i(\omega)$ and $\delta'_r(\omega)$ denote the first derivative with respect to ω of $\delta_i(\omega)$ and $\delta_r(\omega)$, respectively.

A crucial stage in the application of the precedent theorem is to verify that $\delta_r(\omega)$ and $\delta_i(\omega)$ have only real roots. Such a property can be checked while using the following theorem.

Theorem 3 [8]

Let M and N designate the highest powers of s and e^{s} which appear in $\delta^{*}(s)$. Let η be an appropriate constant such that the coefficient of terms of highest degree in $\delta_{r}(\omega)$ and $\delta_{i}(\omega)$ do not vanish at $\omega=\eta$. Then a necessary and sufficient condition that $\delta_{r}(\omega)$ and $\delta_{i}(\omega)$ have only real roots is that in each of the intervals $-2l\pi+\eta<\omega<2l\pi+\eta, l=l_{0}, l_{0}+1, l_{0}+2...$ $\delta_{r}(\omega)$ or $\delta_{i}(\omega)$ have exactly $4lN+M$ real roots for a sufficiently large integer l_{0}.

In this work we will use these theorems in control theory, now we will recall what it was developed in [5, 6, 7] to stabilize a first order delay system where the objective is to determine the stability region in the (K_{p}, K_{i}) parameter space. The K_{p} values are given by the following theorem.

Theorem 4 [5, 6, 8]

The range of K_{p} value, for which a solution to PI stabilization problem for a given stable open-loop plant exists, is given by:

$$-\frac{1}{K}<K_{p}<\frac{T}{KL}\sqrt{\alpha_{1}^{2}+\frac{L^{2}}{T^{2}}} \qquad (7)$$

where α_{1} is the solution of the equation $tan(\alpha)=-\frac{T}{L}\alpha$ in the interval $\left[\frac{\pi}{2}, \pi\right]$.

Now, for a fixed K_{p} inside this range, the integral gain value K_{i} is computing by using the following equation [6, 7]:

$$0<K_{i}<\min_{j=1,3,5...}\left\{a_{j}\right\} \qquad (8)$$

Where $\quad a_{j}=a(z_{j})=\frac{z_{j}}{KL}(\sin(z_{j})+\frac{T}{L}z_{j}\cos(z_{j}))$

and z_{j} present the roots of the imaginary part of closed loop characteristic equation of the system.

4. Stabilization using PI controller for second order delay system

In this section we cheek a solution to PI stabilization problem for second order delay system to demonstrate the results given by theorem 1 in section 2.

The closed-loop characteristic equation of the proposed unity-gain feedback system is given by:

$$\delta(s)=K(K_{i}+K_{p}s)e^{-Ls}+(s^{2}+a_{1}s+a_{0})s \qquad (9)$$

Due to the presence of the exponential term, the above quasi-polynomial has an infinite number of roots, which make the stability analysis of the closed-loop system extremely difficult. Instead, we can use Theorem 2 and 3 to solve this problem.

First we consider the quasi-polynomial $\delta^{*}(s)$:

$$\delta^{*}(s)=e^{Ls}\delta(s)=K(K_{i}+K_{p}s)+s(s^{2}+a_{1}s+a_{0})e^{Ls} \qquad (10)$$

replacing s by $j\omega$, we get:

$$\delta^{*}(j\omega)=\delta_{r}(\omega)+j\delta_{i}(\omega) \qquad (11)$$

with:

$$\begin{cases}\delta_{r}(\omega)=KK_{i}+(\omega^{3}-a_{0}\omega)\sin(L\omega)-a_{1}\omega^{2}\cos(L\omega)\\ \delta_{i}(\omega)=w\left[KK_{p}+(a_{0}-\omega^{2})\cos(L\omega)-a_{1}\omega\sin(L\omega)\right]\end{cases} \qquad (12)$$

Clearly, the parameter K_{i} only affects the real part of $\delta^{*}(j\omega)$ whereas the parameter K_{p} affects the imaginary part.

Let's put $z=L\omega$, we get:

$$\begin{cases}\delta_{r}(z)=KK_{i}+\sin(z)(\frac{z^{3}}{L^{3}}-a_{0}\frac{z}{L})-a_{1}\frac{z^{2}}{L^{2}}\cos(z)\\ \delta_{i}(z)=\frac{z}{L}(KK_{p}+\cos(z)(a_{0}-\frac{z2}{L^{2}})-a_{1}\frac{z}{L}\sin(z))\end{cases} \qquad (13)$$

Step 1.

The application of the second condition of theorem 2, led us to:

$$E(z_{0})=\delta_{i}^{'}(z_{0})\delta_{r}(z_{0})-\delta_{i}(z_{0})\delta_{r}^{'}(z_{0})>0$$

from (13) we get:

$$\delta_{i}^{'}(z)=\frac{KK_{p}}{L}-\sin(z)(a_{0}+\frac{2a_{1}z}{L^{2}}-\frac{z^{3}}{L^{3}})+\cos(z)(\frac{a_{0}}{L}-\frac{3z^{2}}{L^{3}}-a_{1}\frac{z^{2}}{L^{2}})$$

for $z_{0}=0$ (a value that annul $\delta_{i}(z)$) we obtain:

$$E(z_{0})=\delta_{i}^{'}(z_{0})\delta_{r}(z_{0})=(\frac{KK_{p}+a_{0}}{L})KK_{i}>0$$

which implies $K_{p}>\frac{-a_{0}}{K}$ since $K>0$ and $K_{i}>0$.

This proofs the first inequality given by (2) in Theorem 1.

Step 2.

We pass to the verification of the interlacing condition of $\delta_r(z)$ and $\delta_i(z)$ roots. For such purpose, we are going to determine the roots from the imaginary part, which is translated by the following relation:

$$\delta_i(z)=0 \Rightarrow \begin{cases} z=0 \\ or \\ KK_p+\cos(z)(a_0-\frac{z^2}{L^2})-a_1\frac{z}{L}\sin(z)=0 \end{cases}$$

$$\Rightarrow \begin{cases} z=0 \\ or \\ KK_p+\cos(z)(a_0-\frac{z^2}{L^2})=a_1\frac{z}{L}\sin(z) \end{cases}$$

$$\Rightarrow \begin{cases} z=0 \\ or \\ f(z)=g(z) \end{cases}$$

We notice that $z_0=0$ is a trivial root of the imaginary part. The others are difficult to solve analytically. However, this can be made graphically. Two cases are presented:

First case: $\frac{-a_0}{K}<K_p<K_u$

In this case, we graph the curves of $f(z)$, of $g(z)$ and the line h defined by $z=L\sqrt{a_0}$ which are presented in figure 1.

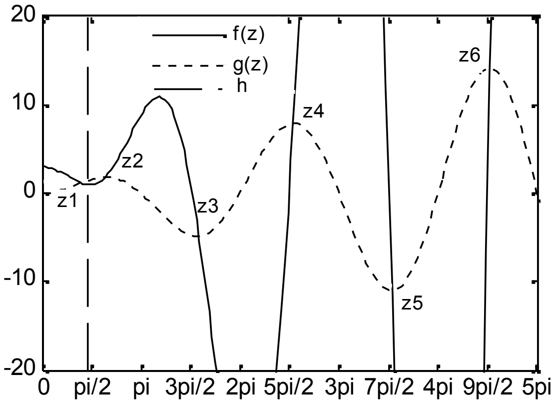

Fig1: Representation of the curves of $f(z)$, of $g(z)$ and of h

(Case: $-\frac{a_0}{K}<K_p<K_u$)

Where K_u is defined in second case. We notice that for $-\frac{a_0}{K}<K_p$ the curve of $f(z)$ intersects the curve of $g(z)$ twice in the interval $[0,\pi]$.

Also we can see the following properties:

$$\begin{cases} z_1\in[0,\pi/2] \\ z_3\in[3\pi/2,2\pi] \\ z_5\in[7\pi/2,4\pi] \\ \vdots \end{cases} and \begin{cases} z_2\in[\pi/2,\pi] \\ z_4\in[5\pi/2,3\pi] \\ z_6\in[9\pi/2,5\pi] \\ \vdots \end{cases}$$

i.e z_j verify $\begin{cases} z_1\in[0,\pi/2] \\ and \\ z_j\in\left[(2j-3)\frac{\pi}{2},(j-1)\pi\right] \quad for\ j\geq 2 \end{cases}$

and we have $\begin{cases} z_1<z=L\sqrt{a_0} \\ and \\ z_j>z=L\sqrt{a_0},\ for\ j\geq 2 \end{cases}$

Second case: $K_p\geq K_u$

Figure 2 represents the case where $K_p=K_u$, and K_u is the maximal value of K_p such as the plots of $f(z)$ and $g(z)$ are tangent in the interval $[0,\pi]$.

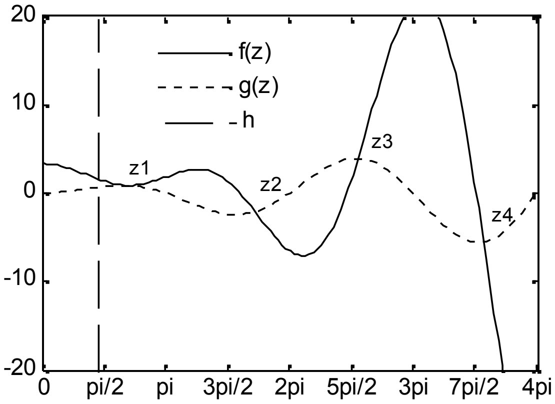

Fig. 2. Representation of the curves of $f(z)$, of $g(z)$ and of h

(Case : $K_p=K_u$)

The plot in Figure 3 corresponds to the case where $K_p>K_u$ and the plot of $f(z)$ does not intersect $g(z)$ in the interval $[0,\pi]$.

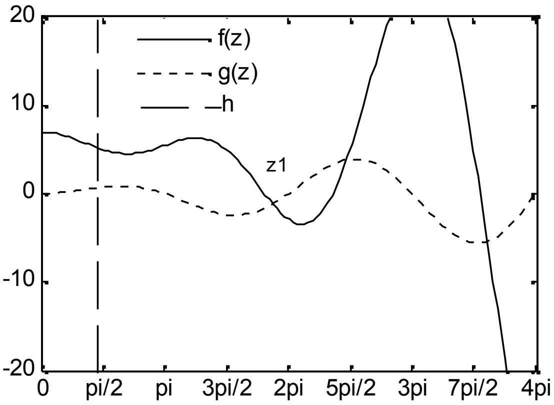

Fig. 3. Representation of the curves of $f(z)$, of $g(z)$ and of h

(Case : $K_p > K_u$)

To verify that $\delta_i(z)$ possess only simple roots we used Theorem 3. Replacing Ls by s_1 in (10), we rewrite $\delta^*(s)$ as follows:

$$\delta^*(s) = e^{Ls}\delta(s) = e^{s_1}\delta(s_1)$$

$$= e^{s_1}\left(\left(\frac{s_1}{L}\right)^3 + a_1\left(\frac{s_1}{L}\right)^2 + a_0\frac{s_1}{L}\right) + K\left(K_p\frac{s_1}{L} + K_i\right)$$

For this new quasi-polynomial, we see that $M = 3$ and $N = 1$ where M and N designate the highest powers of s and e^s which appear in $\delta^*(s)$. We choose $\eta = \frac{\pi}{4}$ that satisfies the condition giving by theorem 3 as $\delta_r(\eta) \neq 0$ and $\delta_i(\eta) \neq 0$. According to figure 1, we notice that for $-\frac{a_0}{K} < K_p < K_u$, $\delta_i(z)$ possess four roots in the interval $\left[0, 2\pi - \frac{\pi}{4}\right] = \left[0, \frac{7\pi}{4}\right]$ including the root at origin. As $\delta_i(z)$ is odd function of z so, it possesses seven roots in $\left[-2\pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}\right] = \left[-\frac{7\pi}{4}, \frac{7\pi}{4}\right]$. Hence, we can affirm that $\delta_i(z)$ has exactly $4N + M = 7$ in $\left[-2\pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}\right] = \left[-\frac{7\pi}{4}, \frac{9\pi}{4}\right]$. In addition, it can be shown that $\delta_i(z)$ has tow real roots in each of the intervals $\left[2l\pi + \frac{\pi}{4}, 2(l+1)\pi + \frac{\pi}{4}\right]$ and $\left[-2(l+1)\pi + \frac{\pi}{4}, -2l\pi + \frac{\pi}{4}\right]$ for $l = 1, 2, \ldots$. It fallows that $\delta_i(z)$ has exactly $4lN + M$ real roots in $\left[-2l\pi + \frac{\pi}{4}, 2l\pi + \frac{\pi}{4}\right]$ for $-\frac{a_0}{K} < K_p < K_u$. At the end, according to theorem 3 $\delta_i(z)$ has only real roots for every K_p in $\left[-\frac{a_0}{K}, K_u\right]$. For $K_p \geq K_u$, corresponding to figure 2 and 3, the roots of $\delta_i(z)$ are not real. We pass to determine the superior value of K_p. According to the definition of K_u, if $K_p = K_u$ then the curves of $f(z)$ and $g(z)$ are tangent in the point α. Which is translated by:

$$\begin{cases} KK_u + \cos(\alpha)\left(a_0 - \frac{\alpha^2}{L^2}\right) = a_1\frac{\alpha}{L}\sin(\alpha) \\ and \\ \frac{d}{dz}\left[KK_u + \cos(z)\left(a_0 - \frac{z^2}{L^2}\right)\right]_{z=\alpha} = \frac{d}{dz}\left[a_1\frac{z}{L}\sin(z)\right]_{z=\alpha} \end{cases}$$

$$\Rightarrow -2\alpha\cos(\alpha)(1 + a_1L) + \sin(\alpha)(\alpha^2 - a_0L^2 - a_1L) = 0$$

$$\Rightarrow \tan(\alpha) = \frac{\alpha(2 + a_1L)}{(\alpha^2 - a_0L^2 - a_1L)} \qquad (14)$$

once α is determined, the parameter K_u is expressed by (15):

$$K_u = \frac{1}{K}\left(a_1\frac{\alpha}{L}\sin(\alpha) - \cos(\alpha)\left(a_0 - \frac{\alpha^2}{L^2}\right)\right) \qquad (15)$$

This completes the proof of the Theorem 1.

After the determination of the roots of imaginary part $\delta_i(z)$, we pass to the evaluation of these roots by real part $\delta_r(z)$.

$$\delta_r(z) = KK_i + \sin(z)\left(\frac{z^3}{L^3} - a_0\frac{z}{L}\right) - a_1\frac{z^2}{L^2}\cos(z)$$

$$= K\left[K_i - a(z)\right] \qquad (16)$$

where $a(z) = \frac{z}{KL}\left[\sin(z)\left(a_0 - \frac{z^2}{L^2}\right) + a_1\frac{z}{L}\cos(z)\right]$.

Let's put z_j, $j = 1, 2, 3\ldots$ the roots of $\delta_i(z)$

for $z_0 = 0$, we have:

$$\delta_r(z_0) = K(K_i - a(0)) = KK_i > 0 \qquad (17)$$

for $z_j \neq z_0$, where $j = 1, 2, 3\ldots$, we get:

$$\begin{aligned}\delta_r(z_j) &= K(K_i - a(z_j)) \\ &= K(K_i - a_j)\end{aligned} \qquad (18)$$

with $a(z_j) = a_j$.

Interlacing the roots of $\delta_r(z)$ and $\delta_i(z)$ is equivalent to $\delta_r(z_0) > 0$ (since $K_i > 0$), $\delta_r(z_1) < 0, \delta_r(z_2) > 0...$

We can use the interlacing property and the fact that $\delta_i(z)$ has only real roots to establish that $\delta_r(z)$ possess real roots too.

From the previous equations we get the following in-equalities:

$$\begin{cases}\delta_r(z_0) > 0 \\ \delta_r(z_1) < 0 \\ \delta_r(z_2) > 0 \\ \delta_r(z_3) < 0 \\ \delta_r(z_4) > 0 \\ \vdots\end{cases} \Rightarrow \begin{cases}K_i > 0 \\ K_i < a_1 \\ K_i > a_2 \\ K_i < a_3 \\ K_i > a_4 \\ \vdots\end{cases} \qquad (19)$$

From these inequalities, it is clear that the a_j odd bounds must be strictly positive; however the a_j even bounds are negative in order to find a feasible range of K_i. From which we have:

$$0 < K_i < \min_{j=1,3,5...}\{a_j\} \qquad (20)$$

On the following, we are interesting to prove that the odd a_j are strictly positive and that the even a_j are negative in order to affirm the relation (20).

From figure 1, the roots z_j of $\delta_i(z)$ verify:

$$\begin{cases} z_1 \in [0, \pi/2] \\ and \\ z_j \in \left[(2j-3)\frac{\pi}{2}, (j-1)\pi\right] \quad for\ j \geq 2\end{cases} \qquad (21)$$

In addition we have:

$$\begin{cases} z_1 < L\sqrt{a_0} \\ and \\ z_j > L\sqrt{a_0},\ for\ j \geq 2\end{cases} \Rightarrow \begin{cases}(a_0 - \frac{z_1^2}{L^2}) > 0 \\ and \\ (a_0 - \frac{z_j^2}{L^2}) < 0,\ for\ j \geq 2\end{cases}$$

Demonstration 1: Let's prove that $a(z_J) > 0$ for $j = 1,3,5,7....$

Step 1: According to (21), for z_1 we have $\cos(z_1) > 0$ and $\tan(z_1) > 0$.

We suppose that:

$$a(z_1) < 0 \Rightarrow \left[\sin(z_1)(a_0 - \frac{z_1^2}{L^2}) + a_1\frac{z_1}{L}\cos(z_1)\right] < 0$$

$$\Rightarrow \sin(z_1)(a_0 - \frac{z_1^2}{L^2}) < -a_1\frac{z_1}{L}\cos(z_1)$$

$$\Rightarrow \tan(z_1)(a_0 - \frac{z_1^2}{L^2}) < -a_1\frac{z_1}{L}$$

$$\Rightarrow \tan(z_1) < \frac{-a_1\frac{z_1}{L}}{(a_0 - \frac{z_1^2}{L^2})} < 0$$

Which is absurd, as a result: $\left[\sin(z_1)(a_0 - \frac{z_1^2}{L^2}) + a_1\frac{z_1}{L}\cos(z_1)\right] > 0 \Rightarrow a(z_1) > 0$.

Step 2: According to (21), we have $\cos(z_j) > 0$ and $\tan(z_j) < 0$ for z_j where $j = 3,5,7,9,...$.

We suppose that

$$a(z_j) < 0 \Rightarrow \left[\sin(z_j)(a_0 - \frac{z_j^2}{L^2}) + a_1\frac{z_j}{L}\cos(z_j)\right] < 0$$

$$\Rightarrow \sin(z_j)(a_0 - \frac{z_j^2}{L^2}) < -a_1\frac{z_j}{L}\cos(z_j)$$

$$\Rightarrow \tan(z_j)(a_0 - \frac{z_j^2}{L^2}) < -a_1\frac{z_j}{L}$$

$$\Rightarrow \tan(z_j) > \frac{-a_1\frac{z_j}{L}}{(a_0 - \frac{z_j^2}{L^2})} > 0$$

Which is absurd, so $\left[\sin(z_j)(a_0 - \frac{z_j^2}{L^2}) + a_1\frac{z_j}{L}\cos(z_j)\right] > 0 \Rightarrow a(z_j) > 0$ for $j = 3,5,7....$

Demonstration 2: Let's prove that $a(z_j) < 0$ for $j = 2,4,6,8....$

For $z_j, j = 2,4,6,8....$ we have $\cos(z_j) < 0$ and $\tan(z_j) < 0$ according to (21),

We suppose that :
$$a(z_j) > 0 \Rightarrow \left[\sin(z_j)(a_0 - \frac{z_j^2}{L^2}) + a_1\frac{z_j}{L}\cos(z_j)\right] > 0$$

$$\Rightarrow \sin(z_j)(a_0 - \frac{z_j^2}{L^2}) > -a_1\frac{z_j}{L}\cos(z_j)$$

$$\Rightarrow \tan(z_j)(a_0 - \frac{z_j^2}{L^2}) < -a_1\frac{z_j}{L}$$

$$\Rightarrow \tan(z_j) > \frac{-a_1\frac{z_j}{L}}{(a_0 - \frac{z_j^2}{L^2})} > 0$$

Which is absurd, consequently
$$\left[\sin(z_j)(a_0 - \frac{z_j^2}{L^2}) + a_1\frac{z_j}{L}\cos(z_j)\right] < 0 \Rightarrow a(z_j) < 0 \quad \text{for}$$
$j = 2,4,6,8....$

Recapitulation: For every K_p in the interval giving by theorem 1, the parameters $a(z_j)$ verify the following conditions:

$$\begin{cases} a(z_j) > 0, \ for \ j = 1,3,5,7... \\ and \\ a(z_j) < 0, \ for \ j = 2,4,6,8... \end{cases} \qquad (22)$$

To determine the set of all stabilizing (K_p, K_i) values for a second order delay system, we propose a procedure summarized in the following algorithm.

Algorithm for determining PI parameters:

1. Choose K_p in the interval suggested by theorem 1 and initialize $j = 1$,
2. Find the roots z_j of $\delta_i(z)$,
3. Compute the parameter a_j associated with the z_j previously founded,
4. Determine the lower and the upper bounds for K_i as follows:
$$0 < K_i < \min_{j=1,3,5...}\{a_j\}$$
5. Go to step 1.

Once the stability domain is determined, the question is what are the optimum parameters of the PI controller which guarantee the good performance of the closed-loop system? In the following, the genetic algorithm is proposed to answer this need.

5. Optimization

Genetic algorithms are powerful stochastic search strategies inspired by the principles of natural selection and natural genetics. In general, the GAs maintain a population of individuals. The population is able to adapt individuals to a given environment by randomized process of selection, crossover and mutation. The environment provides quality information (fitness) for the individuals, and the selection process favors the individuals of higher quality to survive. Thus, during the evolution process the average quality of the population increases, hopefully leading to an optimum solution [4].

Selection: the selection operator produces copies of elements from actual population according to the selection probability of each element. After the selection phase has taken place, crossover and mutation are to be applied.

Crossover: the new population elements are computed in taking into account the elements of old population given by the selection operator and the crossover probability *Pc*.

Mutation: the mutation operator which is characterized by the mutation probability *Pm*, consists to modify some alleles of each element. Mutation introduces random deviations into the population.

GA is applied in various field think to it high potential for global optimization. Attempts in its application to the design problems of control system and identification have been made. Figure 4 presents the optimization principle by genetic algorithm in control problems.

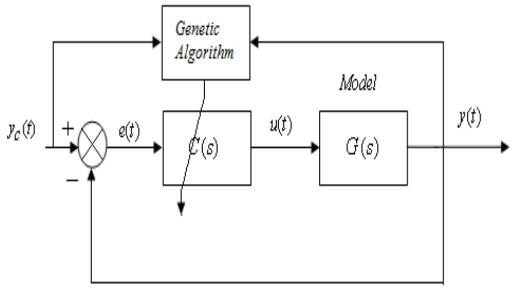

Fig 4: The optimization principle

In our case, we are interest to search the optimum controller parameters in the area of stability using one of following criterion $ISE, IAE, ITAE$ and $ITSE$ described by relation (23):

$$\begin{cases} ISE = \sum_{0}^{t_{\max}} e(t)^2 \\ IAE = \sum_{0}^{t_{\max}} |e(t)| \\ ITAE = \sum_{0}^{t_{\max}} t|e(t)| \\ ITSE = \sum_{0}^{t_{\max}} te(t)^2 \end{cases} \quad (23)$$

If we want to minimize the tuning energy, the ITAE and the IAE criteria are considered. In the case where we privilege the rise time, we take the ITSE criterion. In order to guarantee the tuning energetic cost, we choose the ISE criterion [13].

On the following, the genetic algorithm characterized by generation number equal to 100, $P_c = 0.8$, $P_m = 0.04$ and individual number by population equivalent to 20.

6. Simulation results

To illustrate the application of the results presented in this paper, we consider a high vacuum distillation column where the transfer function between the viscosity $y(s)$ and the reflux flow $u(s)$ is given by [19]:

$$G_1(s) = \frac{y(s)}{u(s)} = \frac{0.0077}{s^2+0.2299s+0.0135} e^{-20s} \quad (24)$$

Using a graphical method estimation applied to the step response of $G_1(s)$, this plant can be expressed by a first order delay system $G_2(s)$ which is described as follows:

$$G_2(s) = \frac{0.59231}{1+14.5136s} e^{-20s} \quad (25)$$

First, we use Theorems 1 and 4 to obtain the range of K_p values over which the sweeping needs to be carried out.

The following table presents the K_p values that stabilize $G_1(s)$ and $G_2(s)$ transfer functions.

TABLE I

K_p VALUES

System	K_p values
$G_1(s)$	$-1.8 < K_p < 2.7968$
$G_2(s)$	$-1.75 < K_p < 3.2$

We sweep over this range of K_p gains and we use the previous algorithm, described in section 4, to determine the range of K_i gains at each step. Figure 5 presents the boundaries obtained in the (K_p, K_i) plane corresponding to $G_1(s)$ and $G_2(s)$ transfer functions.

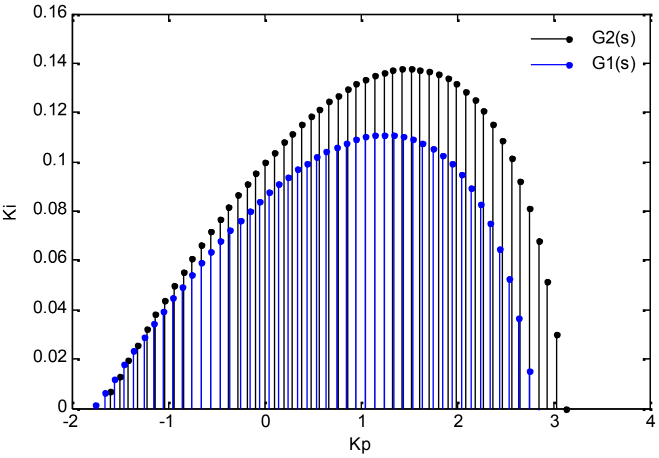

Fig.5: The stabilizing set of (K_p, K_i) values for $G_1(s)$ and $G_2(s)$

To illustrate the influence of stability region on the behavior and on closed-loop response of $G_1(s)$ we set the PI controller parameters to $K_p = 2$ and $K_i = 0.11$. Notice that this point is contained outside of stability region of $G_1(s)$ and inside the stability region of $G_2(s)$ by referring to figure 5. Figure 6 shows the closed-loop responses in the case of considered PI controller.

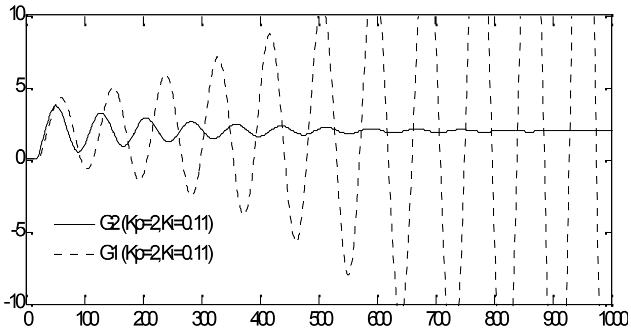

Fig.6: PI ($K_p = 2, K_i = 0.11$) applied to $G_1(s)$ and $G_2(s)$

We remark that this set of values leads to unstable response of $G_1(s)$ and a stable response of $G_2(s)$. In order to quantify the closed-loop system performances, several performance specifications such as ISE, IAE, ITAE and ITSE criteria will be used. We have used GA to obtain optimum PI controller inside stability region.

According to figure 5, the K_p and K_i parameters, considered as population's individuals for the GA, are chosen between [-1.75, 2.79] and [0, 0.11] for $G_1(s)$, and between [-1.6, 3.15] and [0, 0.138] for $G_2(s)$. Optimum PI controller parameters obtained by AG are presented by table II.

TABLE II
OPTIMUM PI PARAMETERS

	Criteria	ISE	IAE	ITAE	ITSE
$G_1(s)$	$K_p opt$	1.3164	0.9391	0.7517	1.0971
	$K_i opt$	0.0422	0.0403	0.0384	0.0425
$G_2(s)$	$K_p opt$	1.4980	1.0955	0.9008	1.2718
	$K_i opt$	0.0507	0.0489	0.0466	0.0512

Table III presents the control law variance in the case of second order delay system $G_1(s)$ using PI controller where the parameters are determined using $G_1(s)$ in case 1 and $G_2(s)$ in case 2.

TABLE III
CONTROL LAW VARIANCE

PI Controllers	ISE	IAE	ITAE	ITSE
Case 1: $G_1(s)$	0.3673	0.1386	0.1144	0.2193
Case 2: $G_2(s)$	0.1527	0.1164	0.1714	0.1129

From Table III, we conclude that the minimal control law variance was generated by the PI-ITAE controller for case 1 and by the PI-ITSE controller for case 2.

Figure 7 shows the closed-loop responses of second order delay system $G_1(s)$ in the case of a PI controller where the parameters are designed using GA approach and $G_1(s)$ model.

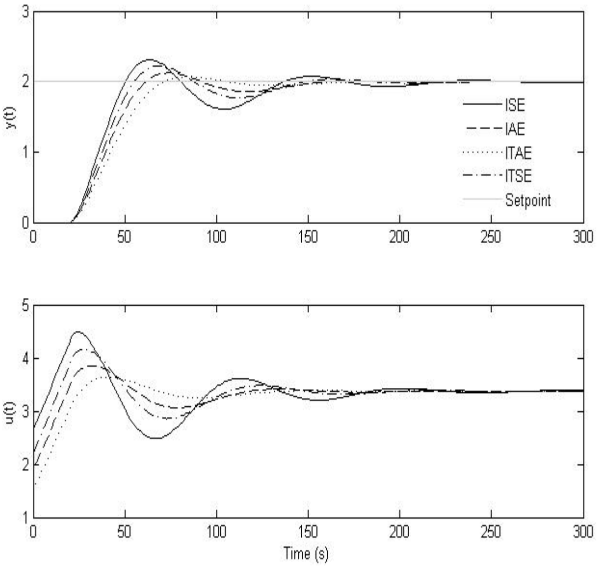

Fig.7: Evolution of the set point, the output and the control signals of $G_1(s)$

(Case 1: PI parameters obtained using GA and $G_1(s)$)

Figure 8 presents the closed-loop responses of second order delay system $G_1(s)$ in the case of a PI controller where the parameters are designed using GA approach and $G_2(s)$ model.

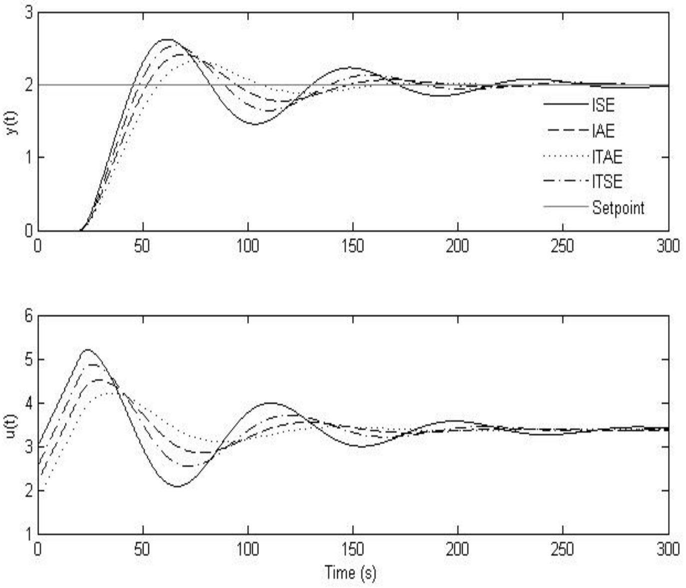

Fig.8: Evolution of the set point, the output and the control signals of $G_1(s)$

(Case 2: PI parameters obtained using GA and $G_2(s)$)

From figures 7 and 8, it is noticed that the outputs of the system suitably follow the set point. Moreover the responses are much oscillatory in case 2. We remark that the minimization of the IAE and ITSE generally produces intermediate responses between fast responses obtained by the minimization of the ISE and slow responses obtained by the minimization of the ITAE.

Table IV shows the comparisons of performance values given by ISE, IAE, ITAE and ISTE methods for second order delay system $G_1(s)$.

TABLE IV
PERFORMANCE CRITERIA

		ISE	IAE	ITAE	ISTE
Case 1: PI designed using GA and $G_1(s)$	ISE	5.0396 e-012	1.8557 e-018	2.1999 e-022	1.2418 e-015
	IAE	5.4614 e-005	3.2279 e-008	3.7184 e-010	9.1182 e-007
	ITAE	0.0509	3.0143 e-005	3.4826 e-007	8.5587 e-004
	ISTE	4.6300 e-009	1.7264 e-015	2.0545 e-019	1.1564 e-012
Case 2: PI designed using GA and $G_2(s)$	ISE	2.0179 e-008	1.0674 e-013	1.1560 e-016	5.5762 e-011
	IAE	0.0039	7.9194 e-006	2.4331 e-007	1.8665 e-004
	ITAE	3.6428	0.0074	2.2613 e-004	0.1742
	ISTE	1.8831 e-005	9.8111 e-011	1.0585 e-013	5.1354 e-008

In this table, the performances' computing of the various criteria is mentioned. Indeed, the first column presents the regulator parameters determined by minimizing ISE criterion using GA; these parameters are used thereafter for calculation of other criteria such as ISE, IAE, ITAE and ITSE. The same procedure of performance computing is reproduced for the other criteria IAE, ITAE and ITSE which are given by the others columns.

Minimization of one of ISE, IAE, ITAE and ISTE, based in the case of GA approach, gives the best performance (the minimal value of the criteria over $t_{\max}$) for ISE criterion. Whereas the maximum performance criteria is provided by ITAE which can explained by the presence of time in criterion.

According to table IV, it is noticed that the corresponding values to case 1 are inferior to those corresponding to case 2 which is explained by the reduction of the stability region on the level of K_i component of the regulator which influences directly the integral of the error.

7. Conclusion

In this work, we have proposed an extension of Hermit-Biehler theorem to compute the stability region for second order delay system controlled by PI controller. The procedure is based first on determining the range of proportional gain value K_p for which a solution to PI stabilization exists. Then, it is shown that for a fixed K_p inside this range, the integral gain value K_i is computed. Lastly, we were interested in search of optimal PI for a given performance criteria (ISE, IAE, ITAE and ITSE), inside the stability region via genetic algorithms. The validation of these results has been done on comparison between first and second order delay systems of vacuum distillation process where we conclude that GA gives acceptable performances.

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