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STOCHASTIC SINGLE MACHINE SCHEDULING WITH RANDOM COMMON DUE DATE

Rachid BENMANSOUR[1], Abdelhakim ARTIBA[1]

Université de Lille Nord
Université de Valenciennes et du Hainaut Cambrésis[1]
LAMIH FRE CNRS 3304 -Le Mont Houy, F-59313
Valenciennes Cedex 9
rachid_emi@hotmail.com ; abdelhakim.artiba@univ-valenciennes.fr

Hamid ALLAOUI[2]

Université de Lille Nord
Université d'Artois[2]
9 rue du Temple - BP 10665 - 62030 Arras Cedex
hamid.allaoui@artois-univ.fr

ABSTRACT: *This paper studies the single machine scheduling problem for minimizing the expected total weighted deviations of completion times from random common due date. Jobs have exponentially distributed processing times and the common due date is a generalized Erlang distribution. The optimal schedules are shown to be* Λ*-shaped. Moreover, we give the optimal schedules when the machine is subject to stochastic breakdowns.*

KEYWORDS: *Stochastic single machine scheduling, common due date, Earliness Tardiness penalties*

1 INTRODUCTION

Scheduling is a decision making that is used on a regular basis in many manufacturing and service industries as well as in most information processing environments. It plays an important role on allowing available resources to tasks over a given period of time and it aims to optimize one or more objectives (Pinedo, 2008). Over the last decade, there is a growing interest in the literature to study scheduling problems for their usefulness in many situations in the real world (transportation, distribution, production). There are two classes of scheduling problems: (a) deterministic scheduling models and (b) stochastic scheduling models. For the latter, the jobs data (processing times, release dates,…) may not be exactly known in advance; only their distribution are known in advance. Interested reader on classes of scheduling models can refer to (Baker and Trietsch, 2009). In the paper of (Lawler et al, 1993), the authors give a detailed overview of deterministic scheduling models whereas (Righter, 1994) does the same thing for stochastic scheduling.

Among the scheduling problems considered within the just-in-time (JIT) concept there are problems in which all the jobs have to be completed as close as possible to a common due date (Gordon et al., 2002). The common due date model corresponds, for instance, to an assembly system in which the components of the product should be ready at the same time.

Scheduling models with both earliness and tardiness are consistent with the JIT philosophy; Goods are produced only when they are needed and jobs are scheduled to complete as close as possible to their due dates (Pinedo, 2008).

The problem of scheduling jobs on machines is usually discussed under the assumptions that the processing times and due dates are known in advance and that the machines are continuously available. However, in practice, some of these assumptions are unrealistic. The scheduler may have to deal with various forms of uncertainties: machines may be subject to lengthy and unpredictable breakdowns and repairs. Processing times and/or due dates may also be of a stochastic nature.

In his paper (Jia, 2001), Jia considered the case where the processing times and the due dates are exponentially distributed random variables and the rate of the due date is an arbitrary positive number. He regarded the total weighted deviations of completion times of jobs about a common due date as objective function. He demonstrates that the optimal schedules are shown to be Λ-shaped. (Pinedo, 1993) and (Pinedo and Rammouz, 1988) studied, under the same assumptions, several regular scheduling problems in which the objective functions are increasing functions of completion times, such as the total weighted tardiness, the maximum lateness or the number of tardy jobs.

In this paper we extend the model presented in (Jia, 2001) in the case when the common due date is any non negative distribution.

To this aim, we use the methodology given in (Elmaghraby et al., 2009) and (Benmansour et al., 2010) to approximate any positive probability density function by a combination of exponential distributions, known as a Phase type distributions (PH-D). Theses distributions were first introduced by Neuts (Neuts, 1975) as the distribution of the absorption time of a finite-state Markov process.

We have comfort that we can accomplish such approximation, since the set of Phase type distributions is dense in the set of non negative distributions (Nelson, 1995). In other words, any non negative continuous distribution can be approximated arbitrarily closely by a PH-D. Further, the set of PH-D's is closed under some operations. In particular, a convex mixture of independent PH-D's is a PH-D, and the convolution of independent PH-D's is a PH-D. A sequence of PH-D with *k* different parameters

$\{\lambda_i\}$ strictly positive (i=1..k) result in a generalized Erlang distribution with k parameters (GED_k). In this paper, we limit our study to GED_2 and GED_3.

2 PROBLEM FORMULATION

We keep the same notation as in (Jia, 2001). Given a set $N=\{1,2,\ldots n\}$, $n>1$, of independent and simultaneously available jobs which are to be processed nonpreemptively on a single machine. Job i ($i \in$ N) requires a non negative random processing time p_i and is assigned a positive weight w_i. D is the common stochastic due date of all jobs.

Let $\boldsymbol{\Pi}$ be the set of all n! possible permutations of the integers 1, 2,…, n, π=([1], [2],…,[n]) $\in \boldsymbol{\Pi}$ which denotes that job [k] is the kth job to be processed on the machine. Assume that there is no inserted idle time in the schedule $\pi \in \boldsymbol{\Pi}$, and the start time of the schedule is 0. Then, under schedule π, the completion time $C_{[i]}$ of job [i] is $C_{[i]} = \sum_{k=1}^{i} p_{[k]}$. The objective here is to find an optimal schedule $\pi^* \in \boldsymbol{\Pi}$ that minimizes:

$$WD(\pi) = E\left(\sum_{i=1}^{n} w_{[i]} \left|C_{[i]} - D\right|\right) \tag{1}$$

$E(X)$ denotes the expectation of random variable X.

3 DETERMINISTIC EQUIVALENT OF THE STOCHASTIC SCHEDULING PROBLEM

Let p_i ($i \in$ N) be exponentially distributed with rate λ_i and D is a GED_2 with parameters a and b strictly positive. Then the probability density function (PDF) of D is given by :

$$f_D(t) = \frac{-ab}{(a-b)}\left(e^{-at} - e^{-bt}\right)$$

and its cumulative distribution function (CDF) by :

$$F_D(t) = 1 + \frac{be^{-at} - ae^{-bt}}{a-b}$$

In order to establish the deterministic equivalent of the stochastic scheduling problem, we first give the following lemma (Jia, 2001).

Lemma. Let X and Y be independent non negative random variables, and Y be exponentially distributed with rate y, then

$$E[\min\{X,Y\}] = \frac{1}{y}\left(1 - E[e^{-yX}]\right).$$

We give in this paper the proof :

Proof :

Let Z be a random variable defined as $Z=min(X,Y)$.

Then the CDF of Z is given by :

$$\begin{aligned} F_Z(t) &= P(Z \le t) = P(\min(X,Y) \le t) \\ &= 1 - P(\min(X,Y) > t) \\ &= 1 - P(X > t)P(Y > t) \\ &= 1 - e^{-yt}\left(1 - P(X \le t)\right) \end{aligned}$$

Therefore

$$f_Z(t) = \frac{dF_Z(t)}{dt} = -ye^{-yt}\left(F_X(t) - 1\right) + e^{-yt} f_X(t)$$

Assuming that F_X, the CDF of random variable X, is derivable.

The expectation of random variable Z is given by:

$$E[Z] = \int_0^\infty t f_Z(t) dt$$

$$E[Z] = \int_0^\infty te^{-yt} f_X(t)dt - y\int_0^\infty te^{-yt} F_X(t)dt + y\int_0^\infty te^{-yt}dt$$

It is easy to derive the following integrals:

$$\int_0^\infty te^{-yt} F_X(t)dt = \int_0^\infty \frac{t}{y} e^{-yt} f_X(t)dt + \int_0^\infty \frac{1}{y^2} e^{-yt} f_X(t)dt$$

$$y\int_0^\infty te^{-yt}dt = \frac{1}{y}$$

And finally derive the expression of E[Z].

Let U be a random variable defined as $U=min(X,D)$.

X is a non negative random variable and D is a GED_2 with parameters a and b. Furthermore X and D are independent.

Then the CDF of U is given by :

$$F_U(t) = 1 - \left(1 - F_X(t)\right)\left(\frac{a}{a-b}e^{-bt} + \frac{b}{b-a}e^{-at}\right)$$

We derive then the PDF of U :

$$\begin{aligned} f_U(t) &= \frac{a}{a-b}\left[e^{-bt} f_X(t) - be^{-bt}\left(F_X(t) - 1\right)\right] \\ &+ \frac{b}{b-a}\left[e^{-at} f_X(t) - ae^{-at}\left(F_X(t) - 1\right)\right] \end{aligned}$$

The expectation of random variable U is given by :

$$\begin{aligned} E[U] &= \int_0^\infty t f_U(t) dt \\ &= \left(\frac{a}{a-b}\right)\int_0^\infty t\left(e^{-bt} f_X(t) - be^{-bt}\left(F_X(t) - 1\right)\right)dt \\ &+ \left(\frac{b}{b-a}\right)\int_0^\infty t\left(e^{-at} f_X(t) - ae^{-at}\left(F_X(t) - 1\right)\right)dt \end{aligned}$$

Using the integral calculated to proof the lemma above, we get:

$$E[\min\{X,D\}] = \frac{a}{b(a-b)}\left(1 - E[e^{-bX}]\right) + \frac{b}{a(b-a)}\left(1 - E[e^{-aX}]\right) \tag{2}$$

This result will allow us to calculate the expected total weighted deviations of completion times:

$$\begin{aligned} E\left(\left|C_{[i]} - D\right|\right) &= E\left(\max\left(C_{[i]} - D, 0\right)\right) \\ &+ E\left(\max\left(D - C_{[i]}, 0\right)\right) \end{aligned}$$

$$E\left(\left|C_{[i]}-D\right|\right)=E\left(C_{[i]}-\min\left(C_{[i]},D\right)\right)$$
$$+E\left(D-\min\left(D,C_{[i]}\right)\right)$$

$$E\left(\left|C_{[i]}-D\right|\right)=E\left[C_{[i]}\right]+E\left[D\right]-2E\left[\min\left\{C_{[i]},D\right\}\right]$$

$$E\left(\left|C_{[i]}-D\right|\right)=\sum_{k=1}^{i}\frac{1}{\lambda_{[k]}}-\left(\frac{1}{a}+\frac{1}{b}\right)$$
$$+\frac{2a}{b(a-b)}\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+b}+\frac{2b}{a(b-a)}\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+a}$$

Then (1) can be written as

$$WD(\pi)=\sum_{i=1}^{n}w_{[i]}\left(\sum_{k=1}^{i}\frac{1}{\lambda_{[k]}}\right)-\left(\frac{1}{a}+\frac{1}{b}\right)\sum_{i=1}^{n}w_{[i]}$$
$$+\frac{2a}{b(a-b)}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+b}\right) \qquad (3)$$
$$+\frac{2b}{a(b-a)}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+a}\right)$$

4 THE Λ-SHAPED PROPERTY OF OPTIMAL SCHEDULES FOR GED$_2$

As a generalization of the total tardiness problem in the deterministic case, one may consider that an earliness penalty is incurred whenever a job, say *j*, is completed before its due date d_j. In this case, the objective function is no more regular. In minimization of the total earliness and the total tardiness with a common due date *d*, it has been shown (Pinedo, 2008) that in an optimal schedule the jobs in set J_1 (contains the jobs that are completed early, i.e $C_j \leq d$) are scheduled first according to longest processing time rule (LPT) and the jobs in J_2 (contains the jobs that are started late) are scheduled last according to shortest processing time rule (SPT). In between these two sets of jobs there may be one job that started early and completed late.

Because of this property, it is often said that the optimal schedule has a V-shape.

In contrast, A Λ-shaped schedule with respect to processing times is that the jobs are arranged in ascending order of their processing times if they are placed before the largest job, but in descending order if placed after it. To prove Λ-shaped optimality, one should show that a schedule with three consecutive jobs *i*, *j* and *k* such that the processing time of job *j* is smaller than the processing times of jobs *i* and *k* cannot be optimal.

Theorem 1. The optimal schedules that minimize (3) are Λ-shaped in terms of $w_i\lambda_i$ ($i \in N$); where $w_i\lambda_i = w_i/E(p_i)$.

Proof.

Let π^* =([1], [2],…[m-1], i, j, k, [m+3],…[n]), be an optimal schedule for minimizing (2), where $1 \leq m \leq n-2$ and jobs i, j and k are three consecutive jobs with $w_i\lambda_i > w_j\lambda_j$ and $w_k\lambda_k > w_j\lambda_j$. Let π_1 =([1], [2],…[m-1], j, i, k, [m+3],…,[n]) and π_2 =([1], [2],…[m-1], i, k, j, [m+3],…,[n]). From (2) we have

$$WD(\pi_1)-WD(\pi^*)=\frac{w_i\lambda_i-w_j\lambda_j}{\lambda_i\lambda_j}+\frac{2a}{a-b}T\frac{w_j\lambda_j-w_i\lambda_i}{(\lambda_i+b)(\lambda_j+b)}$$
$$+\frac{2b}{b-a}T'\frac{w_j\lambda_j-w_i\lambda_i}{(\lambda_i+a)(\lambda_j+a)}$$

Where $T=\prod_{l=1}^{m-1}\frac{\lambda_{[l]}}{\lambda_{[l]}+b}$ and $T'=\prod_{l=1}^{m-1}\frac{\lambda_{[l]}}{\lambda_{[l]}+a}$

Similarly, we can get

$$WD(\pi_2)-WD(\pi^*)=\frac{w_j\lambda_j-w_k\lambda_k}{\lambda_j\lambda_k}+\frac{2a}{a-b}T\frac{\lambda_i\left(w_k\lambda_k-w_j\lambda_j\right)}{(\lambda_i+b)(\lambda_j+b)(\lambda_k+b)}$$
$$+\frac{2b}{b-a}T'\frac{\lambda_i\left(w_k\lambda_k-w_j\lambda_j\right)_i}{(\lambda_i+a)(\lambda_j+a)(\lambda_k+a)}$$

let

$$S=\frac{\left(WD(\pi_1)-WD(\pi^*)\right)\lambda_i\lambda_j}{w_i\lambda_i-w_j\lambda_j}+\frac{\left(WD(\pi_2)-WD(\pi^*)\right)\lambda_j\lambda_k}{w_k\lambda_k-w_j\lambda_j} \qquad (4)$$

then :

$$S=\frac{-2\lambda_i\lambda_j ab\prod_{l=1}^{m-1}\lambda_{[l]}}{a-b}\left(\begin{array}{c}\dfrac{1}{(\lambda_i+b)(\lambda_j+b)(\lambda_k+b)\prod_{l=1}^{m-1}\left(\lambda_{[l]}+b\right)}\\ -\dfrac{1}{(\lambda_i+a)(\lambda_j+a)(\lambda_k+a)\prod_{l=1}^{m-1}\left(\lambda_{[l]}+a\right)}\end{array}\right)$$

For each positive pair (a, b) S_2 is negative.

Since $w_i\lambda_i > w_j\lambda_j$ and $w_k\lambda_k > w_j\lambda_j$, we get $WD(\pi_1)-WD(\pi^*)<0$ or $WD(\pi_2)-WD(\pi^*)<0$ which is contrary to the optimality of π^*. The proof of theorem 1 is complete.

4.1 Single machine scheduling with a window due date

In this case we have a window due date in place of the due date distribution. Recently single machine scheduling with a due date window received attention in the literature (Baker and Scudder, 1990).

Let d_1 and d_2 be the lower due date and the upper due date respectively, and C_i the completion time of job *i*. If $C_i < d_1$, job *i* is regarded as an early job; if $C_i > d_2$, job *i* is regarded as a tardy job. Otherwise the job is on time. Let $k = d_2/d_1$, so $d_2 = kd_1$. In the problem above, we consider the case with due date window: $d_1 = D$ and $d_2 = kD$ ($k>1$). The objective function to minimize is :

$$WD(\pi)=E\left\{\sum_{i=1}^{n}w_{[i]}\left(\max\left(C_{[i]}-kD,0\right)+\max\left(D-C_{[i]},0\right)\right)\right\} \qquad (5)$$

Applying the lemma and using the fact that $E(kD)=kE(D)$, (5) can be written as :

$$\begin{aligned}WD(\pi) &= \sum_{i=1}^{n} w_{[i]}\left(\sum_{k=1}^{i}\frac{1}{\lambda_{[k]}}\right) - k\left(\frac{1}{a}+\frac{1}{b}\right)\sum_{i=1}^{n} w_{[i]} \\ &+ \frac{a}{b(a-b)}\sum_{i=1}^{n} w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+b}\right) + \frac{b}{a(b-a)}\sum_{i=1}^{n} w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+a}\right) \\ &+ \frac{ka}{b(a-b)}\sum_{i=1}^{n} w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+b/k}\right) + \frac{kb}{a(b-a)}\sum_{i=1}^{n} w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+a/k}\right)\end{aligned}$$

Similar to the proof of theorem 1, we have:

Corollary 1. The optimal sequences that minimize (5) are $\wedge$-shaped in terms of $w_i\lambda_i$ ($i \in N$).

4.2 Machine subject to stochastic breakdowns

Single machine stochastic scheduling models incorporating breakdowns and repairs were first studied by Glazebrook (Glazebrook, 1984). Subsequently, many papers have investigated stochastic problems on unreliable or repairable machines; see (Li and Glazebrook, 1998) and (Lee and Lin, 2001).

In (Mittenthal and Raghavachari, 1993), the authors studied a single machine model with Earliness/Tardiness costs and breakdowns. They proved that a schedule of V-shaped is optimal if the sum of squared deviations from a common due date is to be minimized, the breakdown process is Poisson and the repair times are independent and identically distributed.

Later, Zhou and Cai (Zhou and Cai, 1997) examine a stochastic scheduling model with n jobs and a single machine, where the processing times of the jobs are random variables with arbitrary distributions, and the machine is subject to stochastic breakdowns. They derive conditions under which a sequence in non decreasing stochastic order of processing times minimizes the total expected cost, whereas a sequence in non decreasing stochastic order of due dates minimizes the maximum expected cost.

Here we consider the objective in (1) when the machine is subject to stochastic breakdowns. The process on the machine can be a sequence of non negative random vectors $\{U_i, D_i\}_{i=1}^{\infty}$ with U_i representing the i^{th} machine uptime and D_i the i^{th} downtime. We assume that $\{U_i\}_{i=1}^{\infty}$ and $\{D_i\}_{i=1}^{\infty}$ are independent, random variables in sequence $\{U_i\}_{i=1}^{\infty}$ are independent and exponentially distributed with rate α, random variables in sequence $\{D_i\}_{i=1}^{\infty}$ are independent and exponentially distributed with rate β.

To take into account breakdowns that occur while a job is being processed (Birge et al., 1990) we consider both of resume and repeat models. In the resume model, the breakdown merely acts as an interruption, i.e., the job in process can be resumed without loss of prior work as soon as the machine is back in operation. In the repeat model, all prior work on an interrupted job is lost, the job processing has to start from scratch when the machine is back in operation, and the processing time is resampled from the same distribution as before.

Let X_i^1 and X_i^2 be the occupation times of job i ($i \in$ N) on the machine (including the processing time p_i and the downtimes while job i is being processed) for the two models mentioned above. Under the assumptions above, because of the "memoryless" property of exponentially distributed processing times, we obtain that X_i^1 and X_i^2 (i $\in$ N) have the same distribution (Jia, 2001). Let X_i be a random variable with the same distribution as X_i^1 and X_i^2 (i $\in$ N). Then using a result of (Pinedo and Rammouz, 1988), we have (Assuming D is a exponentially distributed with rate d):

$$E[X_i] = \left(1+\frac{\alpha}{\beta}\right)\frac{1}{\lambda_i}$$

And

$$E[e^{-dX_i}] = \frac{\lambda_i}{\lambda_i + d + \alpha d/(\beta + d)}$$

We use these results to calculate (1) when the due date D is a GED$_2$ with parameter a and b.

$$E\left(\left|C_{[i]} - D\right|\right) = E\left[C_{[i]}\right] + E[D] - 2E\left[\min\left\{C_{[i]}, D\right\}\right]$$

From (2) we derive:

$$\begin{aligned}E\left(\left|C_{[i]} - D\right|\right) &= \left(1+\frac{\alpha}{\beta}\right)\sum_{k=1}^{i}\frac{1}{\lambda_{[k]}} - \left(\frac{1}{a}+\frac{1}{b}\right) \\ &+ \frac{2a}{b(a-b)}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]} + b + (\alpha b/\beta + b)}\right) \\ &+ \frac{2b}{a(b-a)}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]} + a + (\alpha a/\beta + a)}\right)\end{aligned}$$

Then (1) can be rewritten as

$$\begin{aligned}WD(\pi) &= \left(1+\frac{\alpha}{\beta}\right)\sum_{i=1}^{n} w_{[i]}\left(\sum_{k=1}^{i}\frac{1}{\lambda_{[i]}}\right) - \left(\frac{1}{a}+\frac{1}{b}\right)\sum_{i=1}^{n} w_{[i]} \\ &+ \frac{2a}{b(a-b)}\sum_{i=1}^{n} w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]} + b + (\alpha b/\beta + b)}\right) \\ &+ \frac{2b}{a(b-a)}\sum_{i=1}^{n} w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]} + a + (\alpha a/\beta + a)}\right)\end{aligned} \tag{6}$$

And S is given by :

$$S=\frac{-2ab\lambda_i\lambda_j\prod_{i=1}^{m-1}\lambda_{[i]}}{a-b}\left(\frac{\left(1+\alpha/(\beta+b)\right)}{b(\lambda_i+B)(\lambda_j+B)(\lambda_k+B)\prod_{i=1}^{m-1}(\lambda_{[i]}+B)}-\frac{\left(1+\alpha/(\beta+a)\right)}{a(\lambda_i+A)(\lambda_j+A)(\lambda_k+A)\prod_{i=1}^{m-1}(\lambda_{[i]}+A)}\right)$$

with

$$B=b\left(1+\frac{\alpha}{\beta+b}\right),\quad A=a\left(1+\frac{\alpha}{\beta+a}\right)$$

To proof that S is negative we consider the function:

$$f(x)=\frac{\left(1+\frac{\alpha}{\beta+x}\right)}{x\prod_{i=1}^{n}\left(\lambda_i+x+\frac{\alpha x}{\beta+x}\right)},\ x>0$$

Since f is a decreasing function, then $\frac{f(b)-f(a)}{a-b}$ is positive and S is negative. Then, using the same reasoning as proof of theorem 1, we have the following theorem :

Theorem 2. The optimal schedules that minimize (6) are Λ-shaped in terms of $w_i\lambda_i$ ($i\in$ N).

5 THE *Λ-SHAPED* PROPERTY OF OPTIMAL SCHEDULES FOR GED$_3$

We will demonstrate as in section 4, that all properties for the optimal schedules of problem described in section 2 holds when the common due date is a GED$_3$ with three parameters a, b and c.
The probability density function of D is given by :

$$f_D(t)=abc\left(\frac{e^{-at}}{(a-b)(a-c)}+\frac{e^{-bt}}{(b-a)(b-c)}+\frac{e^{-ct}}{(c-a)(c-b)}\right)$$

The cumulative distribution function of D is given by :

$$F_D(t)=1-\left(\alpha_1e^{-at}+\alpha_2e^{-bt}+\alpha_3e^{-ct}\right),\quad t\geq 0$$

Where

$$\alpha_1=\frac{bc}{(a-b)(a-c)},\alpha_2=\frac{ac}{(b-a)(b-c)},\alpha_3=\frac{ab}{(c-a)(c-b)}$$

then :

$$E\left(\left|C_{[i]}-D\right|\right)=E\left[C_{[i]}\right]+E\left[D\right]-2E\left[\min\left\{C_{[i]},D\right\}\right]$$

$$E\left(\left|C_{[i]}-D\right|\right)=2\left(\frac{\alpha_1}{a}E[e^{-aC_{[i]}}]+\frac{\alpha_2}{b}E[e^{-bC_{[i]}}]+\frac{\alpha_3}{c}E[e^{-cC_{[i]}}]\right)+E\left[C_{[i]}\right]-E\left[D\right]$$

and (1) becomes :

$$WD(\pi)=\sum_{i=1}^{n}w_{[i]}\left(\sum_{k=1}^{i}\frac{1}{\lambda_{[k]}}\right)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\sum_{i=1}^{n}w_{[i]}+\frac{2\alpha_1}{a}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+a}\right)+\frac{2\alpha_2}{b}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+b}\right)+\frac{2\alpha_3}{c}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+c}\right)$$

Theorem 3. The optimal schedules that minimize (1) when D is a GED$_3$ are Λ-shaped in terms of $w_i\lambda_i$ ($i\in$ N).
Proof.
Under the same arguments of the proof of theorem 1, we calculate the quantity S defined in equation (4) :

$$S=\mathrm{K}\left(\frac{1}{(a-b)(a-c)(\lambda_i+a)(\lambda_j+a)(\lambda_k+a)\prod_{i=1}^{m-1}(\lambda_{[i]}+a)}+\frac{1}{(b-a)(b-c)(\lambda_i+b)(\lambda_j+b)(\lambda_k+b)\prod_{i=1}^{m-1}(\lambda_{[i]}+b)}+\frac{1}{(c-a)(c-b)(\lambda_i+c)(\lambda_j+c)(\lambda_k+c)\prod_{i=1}^{m-1}(\lambda_{[i]}+c)}\right)$$

With

$$\mathrm{K}=-2\lambda_i\lambda_j abc\left(\prod_{k=1}^{m-1}\lambda_{[k]}\right)$$

To proof now that S is negative, its sufficient to proof that the quantity H_n defined as :

$$H_n=\frac{1}{(a-b)(a-c)\prod_{i=1}^{n}(a+\lambda_i)}+\frac{1}{(b-a)(b-c)\prod_{i=1}^{n}(b+\lambda_i)}+\frac{1}{(c-a)(c-b)\prod_{i=1}^{n}(\lambda_i+c)}$$

is positive for all positive numbers a,b,c, λ_1, λ_2, … λ_n.
We suppose that a<b<c (note that S is symmetric in a, b, and c) and we define the function f as :

$$f(x)=\frac{1}{\prod_{i=1}^{n}(x+\lambda_i)},\ x>0,n\in N^*$$

Then

$$H_n=\frac{f(a)}{(a-b)(a-c)}+\frac{f(b)}{(b-a)(b-c)}+\frac{f(c)}{(c-a)(c-b)}$$

$$H_n=\frac{1}{(a-b)(a-c)}\left[f(a)-f(b)\right]+\frac{1}{(c-a)(c-b)}\left[f(c)-f(b)\right]$$

f is continuous function on the closed interval [a, b] and differentiable on the open interval]a, b[, then there exists some α_1 in]a, b[such that :

$$f'(\alpha_1)=\frac{f(a)-f(b)}{a-b}$$

Similarly there exists some α_2 in]b,c[such that :

$$f'(\alpha_2)=\frac{f(c)-f(b)}{c-b}$$

Hence

$$H_n=\frac{1}{c-a}\left[f'(\alpha_2)-f'(\alpha_1)\right]$$

And since

$$f(x)^{''}=f(x)\left(\left(\sum_{k=1}^{n}\frac{1}{x+\lambda_k}\right)^2+\sum_{k=1}^{n}\frac{1}{\left(x+\lambda_k\right)^2}\right)$$

Then the derivative function of *f*, *f'* is strictly increasing for all x in [0,+∞[then :

$$f'(\alpha_2)-f'(\alpha_1)>0$$

And H_n is positive.
The proof of theorem 3 is complete.

5.1 Single machine scheduling with a window due date

As described in section 4.1, $d_2=kd_1$ ($k>1$) and $d_1=D$ is a GED$_3$ with parameters *a*, *b* and *c*. therefore we can write the expression in (5) as :

$$\begin{aligned}WD(\pi)&=\sum_{i=1}^{n}w_{[i]}\left(\sum_{k=1}^{i}\frac{1}{\lambda_{[k]}}\right)-k\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\sum_{i=1}^{n}w_{[i]}\\&+\frac{\alpha_1}{a}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+a}\right)+\frac{\alpha_2}{b}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+b}\right)\\&+\frac{\alpha_3}{c}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+c}\right)+\frac{\alpha_1}{a'}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+a'}\right)\\&+\frac{\alpha_2}{b'}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+b'}\right)+\frac{\alpha_3}{c'}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+c'}\right)\end{aligned}\tag{7}$$

With

$$a'=\frac{a}{k},\quad b'=\frac{b}{k},\quad c'=\frac{c}{k}$$

And the expression in (4) becomes:

$$\begin{aligned}S=-\lambda_i\lambda_j abc\prod_{k=1}^{m-1}\lambda_{[k]}&\left(\frac{1}{(a-b)(a-c)(\lambda_i+a)(\lambda_j+a)(\lambda_k+a)\prod_{i=1}^{m-1}\left(\lambda_{[i]}+a\right)}\right.\\&+\frac{1}{(b-a)(b-c)(\lambda_i+b)(\lambda_j+b)(\lambda_k+b)\prod_{i=1}^{m-1}\left(\lambda_{[i]}+b\right)}\\&\left.+\frac{1}{(c-a)(c-b)(\lambda_i+c)(\lambda_j+c)(\lambda_k+c)\prod_{i=1}^{m-1}\left(\lambda_{[i]}+c\right)}\right)\\-\lambda_i\lambda_j a'b'c'\prod_{k=1}^{m-1}\lambda_{[k]}&\left(\frac{1}{(a'-b')(a'-c')(\lambda_i+a')(\lambda_j+a')(\lambda_k+a')\prod_{i=1}^{m-1}\left(\lambda_{[i]}+a'\right)}\right.\\&+\frac{1}{(b'-a')(b'-c')(\lambda_i+b')(\lambda_j+b')(\lambda_k+b')\prod_{i=1}^{m-1}\left(\lambda_{[i]}+b'\right)}\\&\left.+\frac{1}{(c'-a')(c'-b')(\lambda_i+c')(\lambda_j+c')(\lambda_k+c')\prod_{i=1}^{m-1}\left(\lambda_{[i]}+c'\right)}\right)\end{aligned}$$

S is the sum of two negative members (see proof of theorem 3), then S is negative and since $w_i\lambda_i>w_j\lambda_j$ and $w_k\lambda_k>w_j\lambda_j$, we get $WD(\pi_1)-WD(\pi^*)<0$ or $WD(\pi_2)-WD(\pi^*)<0$ which is contrary to the optimality of π^*. Then we have the following corollary.

Corollary 2. The optimal sequences that minimize (7) are Λ-shaped in terms of $w_i\lambda_i$ ($i\in N$).

5.2 Machine subject to stochastic breakdowns

Suppose the machine is subject to breakdowns as described in section 4.2, then the expression in (1) becomes

$$\begin{aligned}E\left(\left|C_{[i]}-D\right|\right)=&\left(1+\frac{\alpha}{\beta}\right)\left(\sum_{k=1}^{i}\frac{1}{\lambda_{[k]}}\right)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\\&+\frac{2\alpha_1}{a}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+a+(\alpha a/\beta+a)}\right)+\frac{2\alpha_2}{b}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+b+(\alpha b/\beta+b)}\right)\\&+\frac{2\alpha_3}{c}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+c+(\alpha c/\beta+c)}\right)\end{aligned}$$

And

$$\begin{aligned}WD(\pi)=&\left(1+\frac{\alpha}{\beta}\right)\sum_{i=1}^{n}w_{[i]}\left(\sum_{k=1}^{i}\frac{1}{\lambda_{[k]}}\right)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\sum_{i=1}^{n}w_{[i]}\\&+\frac{2\alpha_1}{a}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+a+(\alpha a/\beta+a)}\right)\\&+\frac{2\alpha_2}{b}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+b+(\alpha b/\beta+b)}\right)\\&+\frac{2\alpha_3}{c}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+c+(\alpha c/\beta+c)}\right)\end{aligned}\tag{8}$$

Then S is equal to :

$$S=-2\lambda_i\lambda_j abc\prod_{i=1}^{m-1}\lambda_{[i]}\left(\begin{array}{l}\dfrac{\left(1+\dfrac{\alpha}{\beta+a}\right)^2}{(a-b)(a-c)(\lambda_i+A_1)(\lambda_j+A_1)(\lambda_k+A_1)\prod_{i=1}^{m-1}(\lambda_{[i]}+A_1)}\\ +\dfrac{\left(1+\dfrac{\alpha}{\beta+b}\right)^2}{(b-a)(b-c)(\lambda_i+A_2)(\lambda_j+A_2)(\lambda_k+A_2)\prod_{i=1}^{m-1}(\lambda_{[i]}+A_2)}\\ +\dfrac{\left(1+\dfrac{\alpha}{\beta+c}\right)^2}{(c-a)(c-b)(\lambda_i+A_3)(\lambda_j+A_3)(\lambda_k+A_3)\prod_{i=1}^{m-1}(\lambda_{[i]}+A_3)}\end{array}\right)$$

With

$$A_1=a\left(1+\frac{\alpha}{\beta+a}\right),\quad A_2=b\left(1+\frac{\alpha}{\beta+b}\right)\quad A_3=c\left(1+\frac{\alpha}{\beta+c}\right)$$

As before we consider the following function:

$$f(x)=\frac{\left(1+\dfrac{\alpha}{x+\beta}\right)^2}{\prod_{i=1}^{n}\left(x+\lambda_i+\dfrac{\alpha x}{x+\beta}\right)},\ x>0$$

and we use the mean value theorem on *f* to proof that S is negative.

Theorem 4. The optimal schedules that minimize (8) are $\wedge$-shaped in terms of $w_i\lambda_i$ ($i\in N$).

6 STUDY OF THE *Λ-SHAPED* PROPERTY UNDER THE ASSUMPTION THAT THE DUE DATE IS A GED$_4$

In this partical case, we will show by an example to the contrary that the optimal schedules that minimize (1) are not necessary $\wedge$-shaped in terms of $w_i\lambda_i$.

Assume that the due date is a generalized Erlang distribution with four stages and with parameters *a*, *b*, *c* and *d*. then the probability density function of D is given by :

$$f_D(t)=k_1e^{-at}+k_2e^{-bt}+k_3e^{-ct}+k_4e^{-dt}$$

where :

$$k_1=\frac{-abcd}{(a-b)(a-c)(a-d)},\quad k_2=\frac{-abcd}{(b-a)(b-c)(b-d)},$$
$$k_3=\frac{-abcd}{(c-a)(c-b)(c-d)},\quad k_4=\frac{-abcd}{(d-a)(d-b)(d-a)}$$

and its cumulative distribution function is given by :

$$F_D(t)=1-\left(\frac{k_1}{a}e^{-at}+\frac{k_2}{b}e^{-bt}+\frac{k_3}{c}e^{-ct}+\frac{k_4}{d}e^{-dt}\right)$$

Using the lemma 1 we get :

$$\begin{aligned}E\left(\left|C_{[i]}-D\right|\right)=&\left(\sum_{k=1}^{i}\frac{1}{\lambda_{[k]}}\right)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\\ &+\frac{2k_1}{a^2}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+a}\right)+\frac{2k_2}{b^2}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+b}\right)\\ &+\frac{2k_3}{c^2}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+c}\right)+\frac{2k_4}{d^2}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+d}\right)\end{aligned}$$

then (1) becomes :

$$\begin{aligned}WD(\pi)=&\sum_{i=1}^{n}w_{[i]}\left(\sum_{k=1}^{i}\frac{1}{\lambda_{[k]}}\right)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\sum_{i=1}^{n}w_{[i]}\\ &+\frac{2k_1}{a^2}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+a}\right)+\frac{2k_2}{b^2}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+b}\right)\\ &+\frac{2k_3}{c^2}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+c}\right)+\frac{2k_4}{d^2}\sum_{i=1}^{n}w_{[i]}\left(\prod_{k=1}^{i}\frac{\lambda_{[k]}}{\lambda_{[k]}+d}\right)\end{aligned}$$

Therefore we derive from (4)

$$S=L\left(\begin{array}{l}\dfrac{1}{(a-b)(a-c)(a-d)(\lambda_i+a)(\lambda_j+a)(\lambda_k+a)\prod_{i=1}^{m-1}(\lambda_{[i]}+a)}\\ +\dfrac{1}{(b-a)(b-c)(b-d)(\lambda_i+b)(\lambda_j+b)(\lambda_k+b)\prod_{i=1}^{m-1}(\lambda_{[i]}+b)}\\ +\dfrac{1}{(c-a)(c-b)(c-d)(\lambda_i+c)(\lambda_j+c)(\lambda_k+c)\prod_{i=1}^{m-1}(\lambda_{[i]}+c)}\\ +\dfrac{1}{(d-a)(d-b)(d-c)(\lambda_i+d)(\lambda_j+d)(\lambda_k+d)\prod_{i=1}^{m-1}(\lambda_{[i]}+d)}\end{array}\right)$$

with $L=-2\lambda_i\lambda_j abcd\prod_{i=1}^{m-1}\lambda_{[i]}$

The optimality of the $\wedge$-shape schedules will be proved if the following quantity is positive for all reel positive numbers a, b, c, d, λ_1, λ_2, … λ_n.

$$G_n = \frac{1}{(a-b)(a-c)(a-d)\prod_{i=1}^{n}(a+\lambda_i)} + \frac{1}{(b-a)(b-c)(b-d)\prod_{i=1}^{n}(b+\lambda_i)}$$
$$+ \frac{1}{(c-a)(c-b)(c-d)\prod_{i=1}^{n}(\lambda_i+c)} + \frac{1}{(d-a)(d-b)(d-c)\prod_{i=1}^{n}(\lambda_i+d)}$$

It suffices for us to take n=3, (λ_1, λ_2, λ_3) = (1, 2, 3) and (a, b, c, d) = (1, 2, 3, 4). With these parameters we get $G_3 = -1/105$. Therefore the Λ-shape schedules is not optimal.

7 CONCLUSION

In this paper, we consider scheduling jobs with an exponentially processing times and a random due date.
We show that the single machine scheduling problem with such model is optimally solved by scheduling jobs in a Λ-shape manner. These results remain valid when the machine is subject to stochastic breakdowns with independent and exponentially distributed uptimes $\{U_i\}_{i=1}^{\infty}$ and independent and exponentially distributed downtimes $\{D_i\}_{i=1}^{\infty}$; Furthermore $\{U_i\}_{i=1}^{\infty}$ and $\{D_i\}_{i=1}^{\infty}$ are assumed to be independent.
Most of the case, GED_2 and GED_3 have been shown to give a good approximation of non negative continuous distribution (Benmansour et al., 2010). We show further more that optimality of the Λ-shape schedules is not valid in the case when the due date is a GED_4.
For further research, we will explore the validity of the theorems given above for a single machine subject to stochastic breakdowns with a more general distribution.

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