FAIR AND RATIONAL SCHEMES FOR PAYOFF ALLOCATION IN SUPPLY CHAIN DESIGN PROBLEMS

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ABSTRACT: *Supply chain design problems can be analyzed as cooperative linear production games. The maximal total payoff and the optimal coalition of a "market responsive" supply network can be obtained from the solution of the mixed-variables Linear Programming problem. Then, using duality theory, the "Owen set" can be constructed in order to allocate the payoff among the members of the optimal coalition. However, it is shown that for a classical aggregate planning model, such an allocation scheme may be unfair and its stability critical. Two techniques are proposed to correct this defect. The first one consists in introducing transfer payments among the players of the optimal coalition. Then, as this technique is not always feasible, a second technique is proposed. It relies on an improved representation of capacitated resources through clearing functions. It is shown that a better payoff allocation mechanism can then be computed through an appropriate tuning of clearing functions.*

KEYWORDS: *Supply networks, Game theory, Manufacturing systems, Linear production games, Duality*

1 INTRODUCTION

The concept of a supply chain concentrates some major features of business organization in today's Society. Typically, it represents a network of autonomous production units connected through a communication network carrying manufacturing proposals, products orders and delivery protocols.

Classically, three main stages can be distinguished in the study of a supply chain:

- the design stage, in which products and partners are selected to maximize the potential economic efficiency of the coalition of suppliers, producers and retailers, through professional complementarities and business synergy,
- the negotiation stage, in which partners elaborate contracts defining their commitments and terms of trade
- the operational stage which organizes in real-time the coordination and control of product flows and manufacturing tasks.

These stages differ by their decisional level: strategic, tactic and operational, with their associated time scales, as in classical hierarchical production planning systems (Bitran and Hax, 1977). However, the distributed and hierarchical nature of supply chains modifies the type of models to be used at each stage. Also, decisions taken at upper decisional levels (respectively strategic, tactic) can only serve as frameworks and guidelines for lower decisional levels (respectively tactic, operational).

In the design and negotiation stages, complexity proceeds from two main reasons:

- existence of many possible enterprise coalitions and distribution patterns for tasks and rewards,
- association of partners with different and often conflicting objectives.

Such characteristics are of major concern in game theory, under the classical decomposition into cooperative (or coalitional) and non-cooperative games.

In the light of cooperative game theory, a supply chain can be modelled as a coalition of partners pooling their resources and sharing the same utility function (profit). In this framework, utilities are transferable and the supply chain construction can be analysed as a TU (Transferable Utility)-game. Such a model may not be fully realistic in the sense that it does not capture subjective preferences and autonomy of supply chain partners. However, it can be seen as a valuable limit model to determine the maximal value of the chain and the shares of the global profit objectively acceptable by all the partners. In practice, such shares could then be negotiated and implemented through contracts. The works of Cachon and Netessine (2004) and Nagarajan and Sošić (2008) provide convincing interpretations of supply chain design problems as cooperative games. In this study, the problem of supply network optimal design is obtained from the solution of a linear production game (Shapley and Shubik 1972, Owen 1975). A practical advantage of this model is that it evaluates and compares different possible coalitions and can be used as an argument in the supply chain design stage, to convince the partners to be part of the best possible coalition and to set up a joint venture.

Section 2 introduces some basic results on cooperative games. Section 3 models a supply chain design problem as a cooperative linear production game and proposes an optimal allocation scheme with an illustrative example. Section 4 proposes improving the payoff policy through side payments. A condition is given under which all the members of the winning coalition obtain a positive profit. However, as this property cannot always be achieved, another method is proposed in section 5, to introduce fairness in the payoff policy through reformulation of resource capacity constraints as clearing functions. Some practical conclusions are drawn in section 6.

2 THE LINEAR PRODUCTION GAME

A TU cooperative (or coalitional) game in the sense of Von Neumann and Morgenstern (1944) can be characterized by:

- A finite set of players $\mathcal{N} = \{1,...,N\}$
- A value function (or characteristic function) $v(S) \geq 0$, with $S \in \mathcal{P}(S)$.

The set $\mathcal{P}(S)$ is the set of all the subsets of $\mathcal{N}$. The value $v(S)$ is the maximal utility (or payoff) that can be obtained by the players of coalition S without any help from players that belong to $\mathcal{N}$ but not to S noted $\mathcal{N} \setminus S$. By definition, v(∅)=0. If coalition S, $S \subseteq \mathcal{N}$ is selected, then each player $i \in S$ receives a share $v_i(S) \geq 0$ such that: $\sum_{i \in S} v_i(S) = v(S)$.

A TU-game is noted $(\mathcal{N}, v)$. It raises two basic problems:

- Global utility maximization: determination of the maximal value function, v^*, and of the coalitions S for which this value is obtained. Two particular coalitions achieving $v(S) = v^*$ are considered: the so-called "grand coalition, $\mathcal{N}$, and a coalition (not necessarily unique) with minimal cardinality, S^*satisfying $v(S^*) = v^*$.
- Assignment problem: determination of the endowments of the agents by distributing the global payoff among them.

As in (Osborne and Rubinstein 1994), an S-feasible payoff profile is defined as a vector $(\rho_i)_{i \in S}$ such that $\sum_{i \in S} \rho_i = v(S)$, and a feasible payoff profile as a vector $(u_i)_{i \in \mathcal{N}}$ such that $\sum_{i \in \mathcal{N}} u_i = v(\mathcal{N})$.

Considering a feasible payoff profile $(u_i)_{i \in N}$, we define for any coalition S, $u(S) = \sum_{i \in S} u_i$.

An allocation rule should satisfy certain properties. Some particular sets have been defined to characterize efficiency and stability of coalitions. One of the most remarkable sets is the core of the game, as defined by Gillies (1959).

Definition 1 Core

The core of a TU-game $(\mathcal{N}, v)$ is the set of feasible payoff profiles $(u_i)_{i \in N}$ that satisfies the following proprieties:

1. Efficiency:

$$\sum_{i \in \mathcal{N}} u_i = v(\mathcal{N}) \tag{1}$$

2. Rationality:

$$v(S) \leq u(S) \text{ for every coalition } S. \tag{2}$$

An interesting index to characterize the value of a subset $S \subseteq \mathcal{N}$ is its marginal contribution, denoted $\Delta(S)$ and defined as follows:

$$\Delta(S) = v(\mathcal{N}) - v(\mathcal{N} \setminus S) \tag{3}$$

Property 1

An allocation that lies in the core satisfies the marginal contribution principle:

$$u(S) \leq \Delta(S) \text{ for every coalition } S. \tag{4}$$

The proof of this property is straightforward: a feasible payoff profile $(u_i)_{i \in N}$ that lies in the core satisfies $u(\mathcal{N} \setminus S) \geq v(\mathcal{N} \setminus S)$ *for every coalition S*. Then, from $u(S) = u(\mathcal{N}) - u(\mathcal{N} \setminus S) = v(\mathcal{N}) - u(\mathcal{N} \setminus S)$, one derives $u(S) \leq v(\mathcal{N}) - v(\mathcal{N} \setminus S)$.

Hence, the core consists of all allocations of $v(\mathcal{N})$ among the players such that for each coalition it holds that its players together already get at least as much as they can guarantee themselves by splitting off. From the individual viewpoint, $\Delta(\{i\})$ can be regarded as the maximal payoff that player *i* can expect to gain in the game in the sense that if this player claims more, then it is advantageous for the other players belonging to $\mathcal{N} \setminus \{i\}$ to exclude him from the grand coalition $\mathcal{N}$ and divide the value $v(\mathcal{N} \setminus \{i\})$ among themselves.

Another index of interest is a coalition-marginal contribution defined as follows:

Definition 2 Coalition-marginal contribution

$$\Delta_S(\{i\}) = v(S) - v(S \setminus \{i\}) \text{ with } i \in S. \tag{5}$$

The most critical situation for a player in a coalition is when he is a null payoff player (NPP), defined as follows.

Definition 3 Null Payoff Player (NPP)

A NPP of a coalition S is a player $i \in S$ *such that* $v_i(S) = 0$.

The following property derives from the definitions above.

Property 2

If coalition S has minimal cardinality and $i \in S$ *is an NPP, then* $0 = v_i(S) < \Delta_S(\{i\})$.

Clearly, the reverse condition, $\Delta_S(\{i\}) = 0$ would contradict minimality of coalition *S*.

The possible existence of an NPP in a coalition with minimal cardinality can be used to characterize unfair allocation rules. It is important to note that conditions $\Delta_S(\{i\}) > 0$ and $\Delta(\{i\}) = 0$ may be jointly satisfied and that the fact for an allocation of belonging to the core does not imply fairness.

3 THE SUPPLY CHAIN DESIGN PROBLEM

3.1 Problem presentation

In many practical situations, a supply chain can be viewed as a multistage production system in which the different production stages are performed by different enterprises. Requirements planning models (Baker 1993) can then be used to define and distribute responsibilities and manufacturing orders among the partners. In this view, the product structure supports the enterprise network organization, especially under an extended view of the BOM (Bill of Materials), such as the G-BOM (Generic BOM, (Lamothe et al., 2005)), integrating product families rather than simple products, to represent the context of mass customization. Additionally, multistage production by several producers highly differs from multistage production by a single producer because of the need for negotiation, contracts and higher coordination requirements. It also carries new possibilities in the design stage for selecting partners, sharing resources and rewards. These possibilities precisely constitute the main ingredients of linear production games (Shapley and Shubik 1972, Owen 1975).

Consider a set of final products (or families of products) $i \in \{1, \cdots, g\}$. Typically, the *gozinto* graph describes the product structure and has no cycle. It can then be decomposed into levels : level 0 products are the *g* final products. Then, intermediate and primary products are numbered in the increasing order of their level. The level of product *i*, for *i=g+1,...,n* is the maximal number of stages to transform product i into a final product. Each production stage is supposed to have several input products but only one output product. The BOM technical matrix *G*, is defined as follows: according to a given manufacturing recipe, production of one unit of product *i* requires the combination of components $j \in \{1, \cdots, n\}$ in quantities G_{ji}. It can be noted that under a level-consistent ordering of products, matrix *G* has a simple lower triangular structure (Hennet 2003).

There are *N* enterprises who candidate to be part of the supply chain to be created. Each candidate enterprise is characterized by its production resources: manufacturing plants, machines, work teams, robots, pallets, storage areas.

Let $x = ((x_{ik})$ be the matrix of the quantities of product *i* produced (or obtained by exchange) at firm *k* and $y = (y_1, \cdots, y_n)^T$ be the output vector during a reference period. The components of this matrix and vector are the variables of the design problem. For simplicity, quantities per period (or throughputs) are supposed continuous: $x \in \Re_+^{n \times N}$, $y \in \Re_+^n$. The problem can be formulated in terms of the global throughput vector, denoted ω and related to matrix *x* through the elementary summation relation (6):

$$\omega = x\mathbf{1}, \ \mathbf{1} \text{ being the unit vector of dimension } N. \quad (6)$$

The output throughput vector can be computed from the global throughput vector by the following relation:

$$y = (I - G)\omega, \quad (7)$$

with I the $n \times n$ identity matrix.

From the structure of matrix *G*, matrix $(I - G)$ is regular and matrix $(I - G)^{-1}$ is lower triangular (with 1s on the diagonal) and nonnegative. Then the global throughput vector is nonnegative and expressed as follows:

$$\omega = (I - G)^{-1} y. \quad (8)$$

As in (Van Gellekom et al. 2000), a coalition S is defined as a subset of the set $\mathcal{N}$ of N enterprises with characteristic vector $e_S \in \{0,1\}^N$ such that:

$$\begin{cases} (e_S)_j = 1 & \text{if } j \in S \\ (e_S)_j = 0 & \text{if } j \notin S \end{cases} \quad (9)$$

For the R types of resources considered *(r=1,...,R)*, capacity constraints of coalition S are written:

$$\sum_{i=1}^{n} m_{ri}\omega_i \leq \sum_{j \in S} c_{rj} \quad (10)$$

where c_{rj} is the amount of resource r available for enterprise j, $C=((c_{rj}))\in\Re^{R\times N}$, m_{ri} is the amount of resource r necessary to produce *1* unit of product *i*, $M=((m_{ri}))\in\Re^{R\times n}$.

Assume that a payoff $p_i>0$ is obtained from the sale of one unit of final product *i (i=1,…,g)*, and define the unit price vector $p=[p_1,\cdots,p_n]^T$

3.2 The Value chain

Let $\chi=(\chi_1,\cdots,\chi_R)^T$ be the vector of unit costs for resources. The unit cost for product *i*, γ_i, is assumed to be the same for all the firms possessing the required resources. It is determined by the cost resource requirement for producing one unit of product *i*:

$$\gamma_i=\sum_{r=1}^{R}\chi_r m_{ri}\ . \tag{11}$$

Let $\gamma=(\gamma_1,\cdots,\gamma_n)^T$ be the vector of unit production costs for the products. The set of relations (11) for all the products can be written in vector form:

$$\gamma=M^T\chi\ . \tag{12}$$

Prices may be partly or totally exogenous. Let $p=(p_1\cdots p_n)^T$ be the vector of market prices for the final products.

A necessary and sufficient condition for the global profitability of the supply chain is:

$$p^T y-\gamma^T\omega\geq 0\ , \tag{13}$$

or, using relation (8),

$$[p^T-\gamma^T(I-G)^{-1}]y\geq 0\ . \tag{14}$$

The prices of final products can be supposed fixed and exogenous. The global profitability condition (14) can be considered sufficient for the supply chain to be viable.

More restrictive conditions will now be established to allow for a total decomposability of the multistage manufacturing process. It is now assumed that the prices of intermediate products are negotiated in the network so that each manufacturing stage is profitable. Under this more restrictive requirement, the following inequality should be verified for any product *i*:

$$p_i\geq\gamma_i+\sum_{j=1}^{n}G_{ij}p_j \tag{15}$$

Let $\pi=(\pi_1,\cdots,\pi_n)^T$ be the vector of unit profits for products. The unit profit π_i associated with product *i* is:

$$\pi_i=p_i-\gamma_i-\sum_{j=1}^{n}G_{ij}p_j \tag{16}$$

Condition (15) corresponds to the profitability conditions $\pi_i\geq 0\quad\forall i\in\{1,\cdots,n\}$ that characterize the value chain. These conditions can be gathered into the following condition in vector form:

$$\pi=(I-G)p-\gamma\geq 0\ . \tag{17}$$

Condition (17) may appear unnecessarily restrictive. However, it can be used to fix the transfer prices between products in the network so that each manufacturing stage is intrinsically profitable.

$$\begin{aligned}&\text{Maximize}\quad v=\sum_{i=1}^{n}\pi_i y_i\\&\text{subject to}\quad M(I-G)^{-1}y\leq Ce_S\qquad\qquad (P)\\&\quad y\in\Re_+^n,\ \ e_S\in\{0,1\}^N\end{aligned}$$

In problem (*P*), vector e_S is a vector of binary variables.

Problem (P_S) relates to a particular coalition $S\subseteq\mathcal{N}$. It is defined by the same constraints as (*P*), but for binary vector e_S given.

The optimal total payoff for coalition *S* is denoted *v(S).* The TU-game *(𝒩,v)* defined in this section is a Linear Production game and denoted LPG.

3.3 Global profit maximization

The maximal total payoff, v^*, is obtained from the solution of the mixed variables' Linear Programming problem *(P)*:

$$\begin{aligned}&\text{Maximize}\quad v=<(I-G)p-M^T\chi,y>=\sum_{i=1}^{N}\pi_i y_i\\&\text{subject to}\quad M(I-G)^{-1}y\leq Qe\qquad\qquad (P)\\&\quad y\in\Re_+^n,\ \ e\in\{0,1\}^N\end{aligned}$$

with the cardinality of the coalition as the complementary objective to be minimized:

$$s=\sum_{j=1}^{N}(e)_j\ . \tag{18}$$

In problem (*P*), vector e is a vector of binary variables. By definition, problem (P_S) relates to a particular coalition $S\subseteq\mathcal{N}$. It is defined by the same constraints as (*P*), but with the vector of variables e replaced by the known vector e_S given by (9):

$$\begin{cases}(e_S)_j = 1 & \text{if } j \in S \\ (e_S)_j = 0 & \text{if } j \notin S\end{cases}.$$

$$\text{Maximize } v_S = \sum_{i=1}^{n} \pi_i y_i$$
$$\text{subject to } M(I-G)^{-1} y \leq Q e_S \qquad (P_S)$$
$$y \in \Re_+^n.$$

The TU-game $(\mathcal{N}, v)$ defined in this section is an LPG as defined in Owen (1975). It is an *N*-persons game in which the value $v(S)$ of a coalition S is obtained as the solution of an LP problem, (P_S), defined by the nonnegative production matrix $M(I-G)^{-1}$, the nonnegative resource matrix Q and the nonnegative unit profit vector π.

Several properties can be derived from this definition. In particular, it is super-additive: $v(S) + v(T) \geq v(S \cup T)$ for all disjoint coalitions S and T, and it has a non-empty core (Owen, 1975). Superadditivity implies, in particular, that if $S^\star$ is an optimal coalition, then $v(S) = v^*$ for any coalition S such that $S^* \subseteq S$. In particular, the maximal total payoff, v^* can simply be obtained for the grand coalition, $\mathcal{N}$, by solving problem $\mathcal{P}_{\mathcal{N}}$, which is a standard Linear Program with continuous variables.

The secondary objective can be solved together with the global profit maximization problem by adding a 'small' term $-\varepsilon(\sum_{j=1}^{N}(e_S)_j)$ with $0 < \varepsilon << 1$ to v in the objective function of *(P)*. The modified problem, *(P')*, is formulated as follows:

$$\text{Maximize } \varphi = \sum_{i=1}^{n} \pi_i y_i - \varepsilon(\sum_{j=1}^{N} e_j)$$
$$\text{subject to } M(I-G)^{-1} y \leq Q e_S \qquad (P')$$
$$y \in \Re_+^n,\ e \in \{0,1\}^N$$

The term $-\varepsilon(\sum_{j=1}^{N}(e)_j)$ is said to be 'small enough' if the optimal solutions of *(P)* and *(P')* are identical with respect to the optimal vector y^*. This property can obviously be achieved for any value of ε smaller than a threshold value $\varepsilon_0 > 0$. The optimal coalition of lowest cardinality S^* is then obtained by resolution of problem *(P')*:

$$S^* = \{j \ ; j \in \{1, \cdots, N\},\ e_j^* = 1\}$$

This coalition is supposed not empty. Accordingly, the maximal value function of the TU-game, v^*, previously computed for the grand coalition $\mathcal{N}$, is exactly reinstated from the solution of *(P')* by: $v^* = \varphi^* + \varepsilon(\sum_{j=1}^{N} e_j^*)$.

3.4 The purely competitive payoff policy

Numerical resolution of problem ($\mathcal{P}_{\mathcal{N}}$) solves the global utility maximization problem presented in section 3.3 : it defines the maximal total payoff, v^*, and an optimal output vector y^*.

Then, the optimal coalition of the lowest cardinality S^* is obtained by resolution of problem (P'). The optimal value of the criterion v_{S^*} for problem (P_{S^*}) satisfies: $v_{S^*}^* = v^*$ and also $v_{S^*}^* = v_{\mathcal{N}}^*$ since coalition S^* is optimal and ($P_{\mathcal{N}}$) is a relaxation of (P_{S^*}).

To allocate payoffs among coalition partners, consider the dual of (P_S), denoted (D_S).

$$\text{Minimize } w_S = \sum_{r=1}^{R} q_r(S) z_r$$
$$\text{subject to } (I-G)^{-T} M^T z \geq \pi \qquad (D_S)$$
$$z = (z_1, \cdots, z_R)^T \in \Re_+^R$$

The coefficient of variable z_r in the objective function is the quantity of resource r available for production if coalition S is selected:

$$q_r(S) = \sum_{j=1}^{N}(e_S)_j c_{rj} = \sum_{j \in S} c_{rj}\ . \qquad (19)$$

It can be noted that the set of constraints of (D_S) is the same for any coalition *S*. And since the optimal dual variables $z_r^*(S)$ can be interpreted as shadow prices for resources, they determine a vector of payoffs, the so-called "Owen set" for this TU-game, which is optimal in the context of a purely competitive economy (Van Gellekom et al. 2000). Consider the optimal solution (S^*, y^*) of problem (P). The Owen set, which is here the purely competitive payoff profile $u^* = u(S^*)$, is obtained from the solution (w_S^*, $z^*(S^*)$) of (D_{S^*}) by:

$$u^* = Diag\{e_{S^*}\} C^T z^* \qquad (20)$$

where for any vector *V*, $Diag(V)$ is classically defined as the square matrix with the components of vector V on its diagonal and 0s off-diagonal.

Equivalently to (20),

$$\begin{cases} u_j* = \sum_{r=1}^{R} c_{rj} z_r^* \ \text{ if } \ j \in S* \\ u_j* = 0 \ \text{ if } \ j \notin S* \end{cases} . \qquad (21)$$

It is clear that the payoff of each player equals the value of his resource bundle under the shadow price. Moreover, it has been shown in (Owen, 1975) that this vector of payoffs forms a subset of the core in this production game.

Property 3

The feasible payoff profile $(u_i^*)_{i\in N}$ defined by relations (13) belongs to the core of the Linear Production Game.

However, it will be shown that the Owen set assignment has some drawbacks, in particular a lack of fairness that may induce a corresponding lack of stability against objections to coalition *S**.

3.5 Shortcomings of Owen set as a game-theoretical solution rule

The solution rule proposed by (Owen 1975) and called the Owen set, belongs to the core of the Linear Production Game. It is known that the core is a subset of every stable set. So is the Owen set, which is a subset of the core. However, the core is not necessarily a stable set, and even when it is a stable set, this does not imply stability of the Owen set. In addition to its possible lack of coalitional stability, the Owen set defined by (21) can also be seen in some cases as an unfair allocation rule, as defined in section 2, by not rewarding some resources used in the production process.

These drawbacks mainly arise from the fact that resources in excess at the optimal solution have null shadow prices. Therefore, the owners of such resources are not paid for providing them and their incentive to stay in the coalition can only rely on the rewards from the critical resources which they own, if they own any. Thus, the commitment of a partner only possessing resources which are marginally in excess, may be unstable. He is a Null Payoff Player (NPP): his imputation is null although part of his resources contribute to the global value function. These shortcomings are illustrated by the following example.

3.6 Example

Consider the BOM of the example in (Hennet 03) with two final products (1 and 2), three intermediate products (3,4,5), the unit profit vector $\pi = [22\ \ 25\ \ 0\ \ 0\ \ 0]^T$. Four resources are necessary for the five products at the different manufacturing stages, with the following requirement matrix *M* and technical matrix *G*:

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \end{bmatrix}.$$

Ten enterprises are candidate for partnership in the supply chain. The amounts of the four resources owned by the ten enterprises are represented in the following matrix:

$$C = \begin{bmatrix} 10 & 20 & 15 & 5 & 0 & 10 & 0 & 0 & 10 & 0 \\ 0 & 0 & 15 & 5 & 10 & 0 & 0 & 0 & 0 & 15 \\ 0 & 0 & 15 & 5 & 10 & 5 & 0 & 15 & 20 & 0 \\ 10 & 20 & 0 & 5 & 10 & 5 & 30 & 15 & 0 & 15 \end{bmatrix}.$$

The optimal total payoff and optimal coalition are obtained from the solution of the LP (P) (with the additional term to obtain a coalition of lowest cardinality). The maximal total payoff is $v* = 87.5$, obtained for $y* = [0\ \ 3.5\ \ 0\ \ 0\ \ 0]^{\mathrm{T}}$. It is obtained by the minimal coalition $S* = \{3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\}$ and also, indeed, by any coalition containing *S**.

The associated purely competitive payoff profile (Owen set) is obtained by formula (21) for $z^* = [0\ \ 0\ \ 1.25\ \ 0]$:

$u* = u(S*) = [18.75\ \ 6.25\ \ 12.5\ \ 6.25\ \ 0\ \ 18.75\ \ 25]$.

In this solution, the endowment of partner 7 is null, because if partner 7 is in the coalition, resource 4 is in excess and its shadow price falls to 0. Yet, partner 7 is important to the coalition since for coalition $S* - \{7\}$, the total payoff drops down to 54.68! Its individual marginal contribution in S* is:

$\Delta_{S*}(\{7\}) = v(S*) - v(S*\backslash\{7\}) = 32.82$

The property of not assigning any payoff to the players with resources in excess is inherent in the "Owen set" since this solution rule derives from the duality principle in Linear Programming. In duality theory, a shadow price indicates the value of one additional unit of the resource associated with the corresponding primal constraint. Thus if a dual variable is equal to zero ($z_r* = 0$), this means that the addition of one unit of resource r has no effect on the optimal objective function. Hence the allocations based on the values of optimal dual variables exhibit this property: the players with scarce resources share the total worth of the coalition among them, and the players with excess resources get a null payoff. In light of these shortcomings, the Owen set allocation rule may become unfair and critically unstable because the players with excess resources are indifferent between participating or not.

4 IMPROVING PAYOFF FUNCTIONS WITH TRANSFER PAYMENTS

4.1 Trying to combine fairness with rationality

Due to the criticisms directed at the Owen set as a competitive solution, other practical allocation schemes will be explored to ensure that the maximal payoff is fairly allocated to the players of the LPG game. So, the objective is to find an allocation that is fair and belongs to the core of the LPG. These requirements can be formulated by the following set of constraints:

$$\sum_{i=1}^{n} x_i = v^* \quad ; \tag{22}$$

$$\sum_{j\in S} x_j \geq v(S) \quad \forall S \subset \mathcal{N} \tag{23}$$

$$x_j > 0 \quad \forall j \subset S^*. \tag{24}$$

As shown in the previous example, the optimal payoff may strongly decrease if a player who only owns resources that are globally in excess decides not to participate in the cooperative game. This is why condition (24), denoted 'fairness property', has been added to core requirements (22) and (23); it provides a positive gain to every partner of the optimal coalition. However, the existence of solutions to the set of conditions (22), (23) and (24) is not guaranteed in general. The purpose of this section is precisely to characterize the cases when such solutions exist.

By construction, the core of an LPG is not empty if problem ($P_{\mathcal{N}}$) has a finite optimal solution that is strictly positive, $v(\mathcal{N}) > 0$. This is clear from the fact that an Owen solution exists whenever the dual problem ($D_{\mathcal{N}}$), or equivalently (D_{S^*}), has a solution. Then, in order to explore whether the core of the game contains solutions that also satisfy (24), the following set of imputations can be constructed.

The purely competitive payoff profile $u^* = u(S^*)$ has been defined by (21). The set $\mathcal{N}$ can be decomposed into three disjoint subsets, as follows:

$$\mathcal{N} = S_0{}^* \cup S_1{}^* \cup S_2{}^* \tag{25}$$

with, $S_1{}^* = \{i; i \in S^* \text{ and } u_i{}^* > 0\}$, $S_2{}^* = \mathcal{N} - S^*$.

The cardinals of these sets are respectively denoted $s_0{}^*$, $s_1{}^*$, $s_2{}^*$ with, by assumption, $s_0{}^* \geq 1$, $s_1{}^* \geq 1$. This corresponds to the case when the total payoff is strictly positive and some of the players in the optimal coalition are NPP: their Owen allocation is null.
A new set of imputations, denoted w, is defined as follows:

$$\begin{cases} w_i = u_i{}^* + \alpha = \alpha & \forall i \in S_0{}^* \\ w_i = u_i{}^* - \beta & \forall i \in S_1{}^* \\ w_i = u_i{}^* = 0 & \forall i \in S_2{}^* \end{cases}, \tag{26}$$

With $\alpha \geq 0$, $\beta \geq 0$, $\alpha s_0{}^* = \beta s_1{}^*$. (27)

$$\beta \leq \min_{S\subset\mathcal{N}} \left(\frac{\sum_{i\in S_1} u_i{}^* - v(S)}{s_1} \right) \text{ with } S_1 = S \cap S_1{}^* \text{ and}$$

$$s_1 = card(S_1) \tag{28}$$

$$\beta < \min_{i\in S_1{}^*} u_i{}^* \tag{29}$$

Theorem 1
A set of imputations w that satisfies conditions (26)-(29) belongs to the core of the LPG. Furthermore, there exist strictly positive values of β that satisfy (28)-(29) and define fair imputations if and only if $\overline{\beta} = \min_{S\subset\widetilde{S}} \frac{s_0{}^*}{s_1 s_0{}^* - s_0 s_1{}^*} \left(\sum_{j\in S_1} u_j{}^* - v(S \cup S_2{}^*) \right) > 0$ *with* $\widetilde{S} \subset S^*$ *and* $s_1 s_0{}^* - s_0 s_1{}^* > 0$.

The proof of this theorem can be found in (J-C. Hennet and S. Mahjoub, 2009b).

4.2 Example

In the example of section 2.5, the optimal coalition is $S^* = \{3 \; 4 \; 5 \; 6 \; 7 \; 8 \; 9\}$ and the grand coalition $\mathcal{N}$ can be partitioned as follows: $\mathcal{N} = S_0{}^* \cup S_1{}^* \cup S_2{}^*$ with $S_0 = \{7\}$, $S_1 = \{3,4,5,6,8,9\}$ and $S_2 = \{1,2,10\}$. The total number of coalitions in $\mathcal{N}$ is $2^{10} - 1 = 1023$. However, using theorem 1, only $2^6 - 1 = 63$ coalitions have to be tested to determine the value of $\overline{\beta}$. In this example, $s_0{}^* = 1$, $s_1{}^* = 6$ and condition $s_1 s_0{}^* - s_0 s_1{}^* > 0$ requires $s_0 = 0$. Then, the possible values of s_1 (s_1=1,…,6) generate the $2^6 - 1$ sets for which condition (27) has to be tested. The minimal bound, $\overline{\beta} = 1.25$ is obtained for the set $S^* = \{1,2,6,8,10\}$ and since $\overline{\beta} > \min_{i\in S_1} u_i$, the value $\overline{\beta}$ defines the following imputation which belongs to the core and satisfies the condition of fairness:
$\overline{w} = [0 \;\; 0 \;\; 13.75 \;\; 11.25 \;\; 7.50 \;\; 15 \;\; 7.50 \;\; 10 \;\; 22.50 \;\; 0]$.
The proposed technique has actually constructed the set of core allocations $w(\beta)$ defined by parameter β in the interval $[0 \;\; 1.25]$ and such that:

For $\beta > 0$, the solution $w(\beta)$ belongs to the core and has the property of fairness. The feasible interval for β can be used as a negotiation space by the players of the optimal coalition.

4.3 Discussion

The situation described in the numerical example has only 10 players with resource capacities that are not oversized. It has been shown that, in this particular case, the core is not reduced to a single point and it is possible to construct imputations that are both fair and rational.

In contrast to this situation, numerical experiments show that the core of an LPG game is often reduced to a single point, which is precisely the Owen competitive allocation. This result is not surprising if one realizes that in an open market in which many players have similar abilities and equipment, the core of an LPG tends toward a single point that characterizes the perfectly competitive situation. Convergence of the core to the Owen set of an LPG has been shown and characterized by Owen (1975) and Semet and Zamel (1984) for games in which players are replicated when the number of replications tends to infinity. In a broader context, a well-known result is that, for large numbers of players, the core of the game tends to be the set of competitive allocations (Aumann, 1964).

However, the current industrial and commercial practices show that many techniques can be used to avoid the perfectly competitive situation and maintain good profit margins. Our study has provided some clues for achieving these goals by identifying the key property that generates positive profits: ownership or production of assets that are not available elsewhere. For a company, it can be at this stage of the strategic reasoning that the need of technical progress and innovation comes into play.

5 AN IMPROVED LINEAR PRODUCTION FORMULATION

5.1 Introduction of clearing functions

With respect to the Linear Production Game, the main improvement which appears necessary is to better represent resource capacity constraints. It is a well-established property, known as Little's law in queuing theory, that lead times increase proportionally to WIP (Work In Process). And it is also well established that WIP generates holding costs and should be kept as small as possible. Then, as shown in (Asmundsson et al., 2002), the throughput of a resource should actually be represented as a non linear concave function of its desired throughput, called a clearing function. It can be approximated by a set of linear constraints.

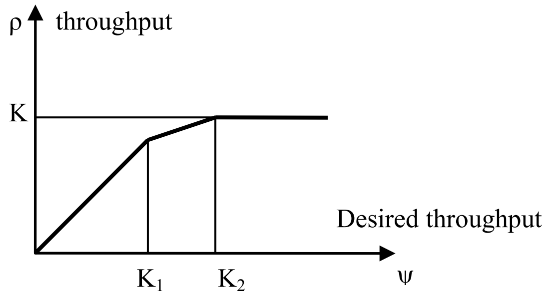

Figure1: A simple piecewise linear clearing function

The simplest type of piecewise linear clearing functions is represented on Fig. 1.

To preserve the sharing of resources, which is a key performance advantage of coalitions with respect to individual firms, clearing functions are applied globally to each aggregated resource of types *r*, $r \in \{1, R\}$. For any coalition *S*, the available quantity of resource r has been defined in (11) and denoted $q_r(S)$. Capacity parameters $q_r^1(S)$ and $q_r^2(S)$ are defined as linear functions of $q_r(S)$:

$$\begin{cases} q_r^1(S) = \alpha_r^1 q_r(S) \\ q_r^2(S) = \alpha_r^2 q_r(S) \end{cases} \text{with } 0 < \alpha_r^1 < 1 < \alpha_r^2 . \qquad (30)$$

As in the initial model, let ω_i be the actual throughput of product *i* and let y_i be the throughput of end product *i*, that is the actual production of product *i* during the reference period. Let ξ_i denote the desired throughput for product *i*. Desired throughputs can be restated as desired productions during the reference period. In the model, quantity $(\xi_i - \omega_i)$ plays the role of WIP for product *i*, with associated unit penalty cost h_i. The modified linear production problem takes the following form.

$$\begin{aligned} &\text{Maximize } v = [\pi^T + h^T (I-G)^{-1}]y - h^T \xi \\ &\text{subject to } M(I-G)^{-1} y \le M\xi \qquad (Q_1) \\ &M(I-G)^{-1} y \le Ce_S \qquad (Q_2) \qquad\qquad (Q) \\ &M(I-G)^{-1} y \le A_1 Ce_S + AM\xi \qquad (Q_3) \\ &\text{with } A_1 = Diag\left\{ \frac{\alpha_1^1(\alpha_1^2 - 1)}{\alpha_1^2 - \alpha_1^1} \;\cdots\; \frac{\alpha_R^1(\alpha_R^2 - 1)}{\alpha_R^2 - \alpha_R^1} \right\}, \\ &A = Diag\left\{ \frac{1-\alpha_1^1}{\alpha_1^2 - \alpha_1^1} \;\cdots\; \frac{1-\alpha_R^1}{\alpha_R^2 - \alpha_R^1} \right\}, \\ &\xi \in \Re_+^n, y \in \Re_+^n,\ e_S \in \{0,1\}^N . \end{aligned}$$

The maximal global payoff, v^* and the optimal coalition, S^*, are obtained as the solution of the mixed linear programming problem, *(Q)*. Then, problem (Q_{S^*}) is derived from problem *(Q)* by replacing the vector of binary variables $e_S \in \{0,1\}^N$ by the characteristic vector of S*,

$e_{S*} \in \{0,1\}^N$. Problem (Q_{S*}) is a standard LP problem in continuous variables $\xi \in \Re_+^n, y \in \Re_+^n$, achieving maximal criterion value *v**. The Owen set related to the Linear Production Game defined by problem *(Q)* can then be obtained from the solution of the dual of (Q_{S*}),noted (Δ_{S*}) and formulated as:

$$\text{Minimize } \Psi_{S*} = \sum_{r=1}^{R} q_r(S*)\left(\zeta_r + \alpha_r^1 \eta_r\right)$$

$$\text{subject to } (I-G)^{-T} M^T(\varphi + \zeta + \eta) \geq p + (I - \Pi)^{-T} h$$

$$M^T(\varphi + A\ \eta) \leq h \qquad (\Delta_{S*})$$

$$\varphi = (\varphi_1, \cdots, \varphi_R)^T \ \in \Re_+^R,$$

$$\zeta = (\zeta_1, \cdots, \zeta_R)^T \ \in \Re_+^R, \eta = (\eta_1, \cdots, \eta_R)^T$$

The new purely competitive payoff profile $u* = u(S*)$, is obtained from the solution ($\Psi_{S*}^*, \varphi^*, \zeta^*, \eta^*$) of ($\Delta_{S*}$) by:

$$u* = Diag\{e_{S*}\}C^T\left(\varsigma^* + A_1\eta^*\right) \qquad (31)$$

It is interesting to note that criterion Ψ_{S*} is a sum of nonnegative terms related to resources. And thus, the payoff profile (31) is efficient in the sense of Definition 1. In fact, this payoff profile is the Owen set associated with the modified Linear Production Game defined by problem *(Q)*.

Shaping the Owen set of the modified linear production game can then be used for providing a positive payoff to the NPPs of the original LPG.

5.2 Example

For simplicity, we select the numerical values $\begin{cases} \alpha_r^1 = 0.25 & \forall r \in \{1, R\} \\ \alpha_r^2 = 3.25 & \forall r \in \{1, R\} \end{cases}$. Then, we obtain $A_1 = A = 0.25I$, where I is the identity matrix of dimension $N \times N$. Vector *p* has the same value as previously, and the unit holding cost vector is: $h = [0.01\ 0.01\ 0.01\ 0.01\ 0.01]^T$. Again, the optimal solution is characterized by $y* = [0\ \ 3.5\ \ 0\ \ 0\ \ 0]^T$, $S* = \{3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\}$, and the optimal value of the new criterion is: $v* = 85.975$. The dual solution, $\Psi_{S*}^* = v* = 85.975$ is obtained for $\zeta^* = [0\ \ 0\ \ 1.22\ \ 0]$, $\varphi^* = [0\ \ 0\ \ 0\ \ 0]$, $\eta^* = [0\ \ 0.01\ \ 0.01\ \ 0.02]$ and the optimal payoff profile is computed by formula (31):

$u(S*) = [18{,}375\ \ 6.15\ \ 12.3\ \ 6.1375\ \ 0.15\ \ 18.4125\ \ 24.45]$. The use of the nonlinear clearing function has improved the stability of coalition S* by rewarding each player in the optimal coalition with a strictly positive endowment.

6 CONCLUSIONS

In this study, we have analyzed the problem of supply chain design through cooperative game approach. For this purpose, a supply chain has been modelled as a coalition of partners pooling their resources and sharing the same payoff function. Due to the problem formulation in the form of a linear production game, linear programming has been used to generate the maximal profit and the optimal coalition. In addition, we have characterized the "Owen set" allocation method and shown that it may become unfair and cause the coalition stability to be insufficiently robust. This behaviour directly derives from the use of linear programming duality theory to determine the payoff of each player. According to this method, each partner receives a profit that equals the value of his resource bundle under the shadow price. In order to ensure that the total payoff is better shared by all the coalition partners, we have proposed two techniques. The first one explores the core of the game to construct a fair solution. But the core is often reduced to the competitive allocation payoff. This is why an alternative technique has been proposed. It consists in modifying the formulation of the production game by introducing piecewise linear clearing functions to better represent resource capacity constraints. By appropriately tuning the clearing function parameters, the coalition stability has been improved through insuring that all the coalition partners receive a strictly positive imputation.

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